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Singular Liouville fields and spiky strings in $\mathbb{R}^{1,2}$ and $\mathrm{SL}(2,\mathbb{R})$

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Abstract: The closed string dynamics in $\mathbb{R}^{1,2}$ and $\mathrm{SL}(2,\mathbb{R})$ is studied within the scheme of Pohlmeyer reduction. In both spaces two different classes of string surfaces are specified by the structure of the fundamental quadratic forms. The first class in $\mathbb{R}^{1,2}$ is associated with the standard lightcone gauge strings and the second class describes spiky strings and their conformal deformations on the Virasoro coadjoint orbits. These orbits correspond to singular Liouville fields with the monodromy matrixes $\pm\, I$. The first class in $\mathrm{SL}(2,\mathbb{R})$ is parameterized by the Liouville fields with vanishing chiral energy functional. Similarly to $\mathbb{R}^{1,2}$, the second class in $\mathrm{SL}(2,\mathbb{R})$ describes spiky strings, related to the vacuum configurations of the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset model.

Keywords: Conformal Field Models in String Theory, Integrable Field Theories, Bosonic Strings, AdS-CFT Correspondence

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Contents

1 Introduction

Integrability of string dynamics in AdS spaces is one of the most actively discussed topic of the last decade due to its important role in the AdS/CFT correspondence. String equations have the most simple and symmetric form in conformal coordinates. The conformal gauge freedom of string dynamics in $AdS \times S$ can be fixed by turning the spherical part of the Virasoro constraint to a positive constant h [1]. Then, the AdS part of the constraint becomes $-h$ and the string description splits in two schemes of Pohlmeyer reduction [2]. They lead to generalized sin-Gordon and sinh-Gordon equations for the sphere and AdS space, respectively. These equations allow a Lax pair representation, which is a basis for the integration of string dynamics. Details of this approach and the related list of references one can find in [3] and [4].

If strings propagate only in AdS space (the case $h = 0$), the described gauge fixing procedure fails. However, the Pohlmeyer scheme can be modified in a conformally invariant form [5–7]. Recently, this approach was effectively used in AdS_3, providing there a new set of interesting string solutions [8, 9].

The Pohlmeyer scheme is usually formulated in terms of a linear system of differential equations for a basis along the string surface. This basis is formed by the tangent and orthogonal vectors to the surface and, therefore, the equations of the linear system contain components of the first and the second fundamental quadratic forms. The consistency conditions for the linear system provide dynamical equations and chirality relations for the worldsheet variables.

In ref. [10] we studied general aspects of Pohlmeyer reduction for AdS strings in arbitrary dimensions. Motivated by the Alday-Maldacena conjecture [11], we have extended the Pohlmeyer scheme to the spacelike surfaces, where the chiral conditions are replaced by holomorphic ones. To simplify the discussion, we turned the worldsheet chiral (or holomorphic) functions to constants. Locally this is always allowed due to the conformal freedom. However, information about some nontrivial string worldsheets might be encoded in global properties of chiral or holomorphic functions (as in [9]) and, therefore, such solutions could be lost by a simple gauging.

Gauge fixing in string theory is a subtle problem even in flat spacetime. It appears that the standard light cone gauge string surfaces in 3d Minkowski space are singular. Namely, the induced metric tensor is degenerated at some points or lines of the surface, and the scalar curvature diverges there.

Among the new solutions constructed in [8] there are the spiky strings [12], which become important ingredients in AdS/CFT correspondence [13–15]. Note that the spiky singularities also correspond to the degeneracy of the induced metric tensor.

A natural question related to non regular string surfaces is to understand the character of singularities and their role in quantized string theory. In the present paper we study this problem for timelike closed strings in three dimensions.

We start with the analysis of the flat case. Using a gauge fixing for the components of the fundamental quadratic forms, we integrate the linear system of the Pohlmeyer scheme and realize that the obtained surfaces are associated with the standard lightcone gauge strings. Then we shown that the chiral $u(z)$ and the antichiral $\bar{u}(\bar{z})$ components of the second quadratic form do not have fixed signs in the lightcone gauge.

To analyze the sector with fixed signs of $u(z)$ and $\bar{u}(\bar{z})$ we use the gauge, which turns these functions to constants $u(z) \mapsto \lambda$, $\bar{u}(\bar{z}) \mapsto \bar{\lambda}$. The consistency condition in this gauge reduces to the Liouville equation, and the periodicity of closed string worldsheets fixes the monodromy class of Liouville fields by the matrixes $\pm I$. These are singular Liouville fields, which are parameterized by the Virasoro coadjoint orbits of the vacuum configurations $T(z) = -n^2/4$ and $\bar{T}(\bar{z}) = -\bar{n}^2/4$, where n and $\bar{n}$ are nonzero integers [16]. The vacuum configurations of the Liouville field describe oscillating circular and rotating spiky strings.

The shape of a spiky string configuration at a fixed time essentially depends on the relative sign between λ and $\bar{\lambda}$, as well as on the values of n and $\bar{n}$. The Virasoro coadjoint orbits define internal degrees of freedom of spiky strings, providing their 'conformal deformations'.

In section 3 we consider strings in AdS_3. Here we 'improve' the integrability of string dynamics by adding the WZ-term to the action, which turns the system to the SL$(2,\mathbb{R})$ WZW theory with Virasoro constraints [17]. This model, called SL$(2,\mathbb{R})$ or $SU(1,1)$ string, was a subject of intensive study in the 90's (see [1] for references).

We apply again the Pohlmeyer type scheme. Using the isometry between $sl(2,\mathbb{R})$ and $\mathbb{R}^{1,2}$, the scheme is formulated in a form equivalent to 3d Minkowski case, which allows to use some results of the previous classification here. Namely, the Kac-Moody currents have the same parameterization as the worldsheet tangent vectors in $\mathbb{R}^{1,2}$. The next step is the integration of the Kac-Moody currents to string worldsheets embedded in SL$(2,\mathbb{R})$. The main difference with the flat case arises just at this point. Another difference is related to the calculation of the worldsheet metric, which involves the string solution, and not only the chiral currents, as in $\mathbb{R}^{1,2}$.

Finally, we summarize the results and discuss open problems. Some technical details are shifted to the appendixes.

2 Closed strings in $\mathbb{R}^{1,2}$

In this section we analyze string dynamics in 3d Minkowski space $\mathbb{R}^{1,2}$ within the scheme of Pohlmeyer reduction [2]. This approach sheds new light to known results of string theory in flat spacetime.

The scalar product in $\mathbb{R}^{1,2}$ we denote by $X\cdot X = X_1\,X_1 + X_2\,X_2 - X_0\,X_0$, where X_0, X_1, X_2 are the coordinates of $X\in\mathbb{R}^{1,2}$. We consider a closed string with periodic boundary conditions $X(\tau,\sigma+2\pi) = X(\tau,\sigma)$. The derivatives with respect to the lightcone coordinates $z=\tau+\sigma$, $\bar{z}=\tau-\sigma$ are denoted by $\partial\equiv\partial_z$, $\bar{\partial}\equiv\partial_{\bar{z}}$.

2.1 Pohlmeyer scheme

The Pohlmeyer scheme for string dynamics is applied in the conformal gauge

$$\partial X\cdot\partial X = 0 = \bar{\partial}X\cdot\bar{\partial}X\,, \tag{2.1}$$

where $X(\tau,\sigma)$ satisfies the free field equation

$$\bar{\partial}\partial X = 0\,. \tag{2.2}$$

The conformal gauge conditions (2.1) provide the relation $\partial_\tau X\cdot\partial_\tau X = 2\partial X\cdot\bar{\partial}X$. The timelikeness of the string worldsheet implies $\partial_\tau X\cdot\partial_\tau X<0$ and, therefore, the non zero component of the induced metric tensor can be parameterized by

$$\partial X\cdot\bar{\partial}X = -e^{\alpha}\,. \tag{2.3}$$

However, it has to be noted that the induced metric can be degenerated ($\partial X\cdot\bar{\partial}X=0$) at some points or lines of the string worldsheet, where the tangent vector $\partial_\tau X$ becomes lightlike. These singular points correspond to $\alpha \to -\infty$, though the functions $X_\mu(\tau,\sigma)$ ($\mu = 0,1,2$) remain smooth (differentiable) there.

To follow the Pohlmeyer scheme, we introduce a basis (B, $\bar{B}$, N) in $\mathbb{R}^{1,2}$ formed by the vectors $B = \partial X$, $\bar{B} = \bar{\partial} X$ and N, which is an unit vector orthogonal to the string surface

$$N \cdot N = 1\,, \qquad\qquad B \cdot N = 0 = \bar{B} \cdot N\,. \tag{2.4}$$

Then, from (2.1)–(2.4) one finds the following linear system of equations for the 'moving' basis along the string world sheet

$$\begin{aligned}
&\partial B = \partial\alpha\, B + u\, N\,, && \bar{\partial} B = 0\,,\\
&\partial \bar{B} = 0\,, && \bar{\partial}\bar{B} = \bar{\partial}\alpha\, \bar{B} + \bar{u}\, N\,,\\
&\partial N = e^{-\alpha}\, u\, \bar{B}\,, && \bar{\partial} N = e^{-\alpha}\, \bar{u}\, B\,,
\end{aligned} \tag{2.5}$$

where u and $\bar{u}$ are the components of the second fundamental form on the worldsheet

$$u = \partial^2 X \cdot N\,, \qquad\qquad \bar{u} = \bar{\partial}^2 X \cdot N\,. \tag{2.6}$$

The consistency conditions for the linear system (2.5) are

$$\begin{aligned}
&\bar{\partial}\partial\alpha + e^{-\alpha}\, u\, \bar{u} = 0\,,\\
&\partial\bar{u} = 0\,, \qquad\qquad \bar{\partial} u = 0\,.
\end{aligned} \tag{2.7}$$

All these equations are invariant under the conformal transformations

$$e^{\alpha(z,\bar{z})} \mapsto \zeta'(z)\bar{\zeta}'(\bar{z})\, e^{\alpha\left(\zeta(z),\bar{\zeta}(\bar{z})\right)}\,, \tag{2.8}$$

$$u(z) \mapsto \zeta'^2(z)\, u(\zeta(z))\,, \qquad\qquad \bar{u}(\bar{z}) \mapsto \bar{\zeta}'^2(\bar{z})\, \bar{u}\left(\bar{\zeta}(\bar{z})\right)\,, \tag{2.9}$$

together with $X(z,\bar{z}) \mapsto X\left(\zeta(z),\bar{\zeta}(\bar{z})\right)$ and $N(z,\bar{z}) \mapsto N\left(\zeta(z),\bar{\zeta}(\bar{z})\right)$. The functions $\zeta(z)$ and $\bar{\zeta}(\bar{z})$ here are monotonic and they satisfy the monodromy conditions

$$\zeta(z+2\pi) = \zeta(z) + 2\pi\,, \qquad \bar{\zeta}(\bar{z}+2\pi) = \bar{\zeta}(\bar{z}) + 2\pi\,. \tag{2.10}$$

This symmetry is a remnant of the reparameterization invariance of string theory in the conformal coordinates. One can use this invariance to remove remaining non physical degrees of freedom and simplify the integration procedure.

We follow this scheme in the next two subsections. The functions $u(z)$ and $\bar{u}(\bar{z})$ are assumed smooth and periodic, like the components of the tangent vectors $B_\mu(z)$ and $\bar{B}_\mu(\bar{z})$. We specify two different classes of $u(z)$, $\bar{u}(\bar{z})$. The first class is formed by the functions which change signs in the interval of periodicity, whereas $u(z)$ and $\bar{u}(\bar{z})$ have no zeros for the second class. The gauge fixing conditions for these classes are different. After integration of the linear system (2.5) we realize that the first class corresponds to the standard lightcone gauge and the second class describes spiky and oscillating circular strings.

2.2 Lightcone gauge

The scheme proposed in this subsection is quite similar to the one used in [18] and [8]. Before integration of the linear system (2.5) we have to find solutions of the consistency conditions (2.7). These conditions are satisfied by the following simple parameterization

$$u(z) = f'(z)\,, \qquad \bar{u}(\bar{z}) = -\bar{f}'(\bar{z})\,, \qquad e^{\alpha} = \frac{1}{2}\,\left[\bar{f}(\bar{z}) - f(z)\right]^2. \tag{2.11}$$

The aim is to describe the class of functions f and $\bar{f}$, which lead to periodic $X(\tau,\sigma)$. Note that solutions of the linear system (2.5) with periodic coefficients, in general, are only quasi-periodic. Therefore, the periodicity of the functions u, $\bar{u}$ and e^{α} is a necessary, but not a sufficient condition for periodicity of $X(\tau,\sigma)$.

The integration of the linear system (2.5) with u, $\bar{u}$ and α given by (2.11) is done in appendix A and it leads to

$$B = f(z)\,\mathbf{e} + \mathbf{e}_+ + f^2(z)\,\mathbf{e}_-\,, \qquad\qquad \bar{B} = \bar{f}(\bar{z})\,\mathbf{e} + \mathbf{e}_+ + \bar{f}^2(\bar{z})\,\mathbf{e}_-\,, \tag{2.12}$$

$$N = \frac{\bar{f}(\bar{z}) + f(z)}{\bar{f}(\bar{z}) - f(z)}\,\mathbf{e} + \frac{2}{\bar{f}(\bar{z}) - f(z)}\,\mathbf{e}_+ + \frac{2\,f(z)\,\bar{f}(\bar{z})}{\bar{f}(\bar{z}) - f(z)}\,\mathbf{e}_-\,. \tag{2.13}$$

Here $\mathbf{e}$, $\mathbf{e}_+$, $\mathbf{e}_-$ are $(z,\bar{z})$-independent $\mathbb{R}^{1,2}$-valued vectors, which arise as integration constants in solving (2.5). The map from the vectors $(\mathbf{e},\mathbf{e}_+,\mathbf{e}_-)$ to $(B,\bar{B},N)$ is invertible and the orthonormality conditions for the basis $(B,\bar{B},N)$ is equivalent to

$$\mathbf{e}\cdot\mathbf{e} = 1\,, \qquad \mathbf{e}\cdot\mathbf{e}_+ = 0 = \mathbf{e}\cdot\mathbf{e}_- = \mathbf{e}_+\cdot\mathbf{e}_+ = \mathbf{e}_-\cdot\mathbf{e}_-\,, \qquad \mathbf{e}_+\cdot\mathbf{e}_- = -\frac{1}{2}\,. \tag{2.14}$$

These conditions are realized by

$$\mathbf{e}_{\pm} = \frac{1}{2}\,(\mathbf{e}_0 \pm \mathbf{e}_1)\,, \qquad \mathbf{e} = \mathbf{e}_2\,, \tag{2.15}$$

where $\mathbf{e}_\mu$ $(\mu = 0,1,2)$ is the standard orthonormal basis in $\mathbb{R}^{1,2}$, with $\mathbf{e}_\mu^{\ \nu} = \delta_\mu^{\ \nu}$. The scalar and exterior products of these basis vectors

$$\mathbf{e}_\mu\cdot\mathbf{e}_\nu = \eta_{\mu\nu} = \mathrm{diag}(-1,1,1)\,, \qquad \mathbf{e}_\mu\times\mathbf{e}_\nu = \epsilon_{\mu\nu}^{\ \ \ \rho}\,\mathbf{e}_\rho\,, \tag{2.16}$$

are given by the metric $\eta_{\mu\nu}$ and the Levi-Civita $\epsilon_{\mu\nu\rho}$ tensors, respectively. With $\epsilon_{012} = 1$, the algebra of exterior products yields $\mathbf{e}\times\mathbf{e}_\pm = \pm\,\mathbf{e}_\pm$ and $2\mathbf{e}_-\times\mathbf{e}_+ = \mathbf{e}$. Using then (2.11), the normal vector (2.41) can be written in a Lorentz invariant form

$$N = e^{-\alpha}\,\bar{B}\times B\,. \tag{2.17}$$

Other realizations of (2.14) are obtained by Lorentz transformations of (2.15).

The tangent vectors to a closed string worldsheet $B = \partial X$ and $\bar{B} = \bar{\partial} X$ are periodic chiral and antichiral vector functions respectively. Hence, the functions $f(z)$ and $\bar{f}(\bar{z})$ in (2.12) have to be periodic as well and they enjoy the Fourier mode expansions

$$f(z) = p + \sum_{n\neq 0} a_n\,e^{-inz}\,, \qquad \bar{f}(\bar{z}) = \bar{p} + \sum_{n\neq 0} \bar{a}_n\,e^{-in\bar{z}}\,. \tag{2.18}$$

In addition, the periodicity of $X(\tau,\sigma)$ requires equality of the zero modes of $B(z)$ and $\bar{B}(\bar{z})$. These conditions are given by

$$p = \bar{p}\,, \qquad\qquad \sum_{n>0} |a_n|^2 = \sum_{n>0} |\bar{a}_n|^2\,. \tag{2.19}$$

Eqs. (2.18)–(2.19) define the class of $f(z)$, $\bar{f}(\bar{z})$ leading to periodic $X(\tau,\sigma)$. These string surfaces, besides $f(z)$ and $\bar{f}(\bar{z})$, depend on the parameters of Lorentz transformations of the basis (2.15) and also on three integration constants related to the final equations $\partial X = B$ and $\bar{\partial} X = \bar{B}$.

The general solution of the consistency conditions (2.7), given by [8]

$$e^{\alpha} = \frac{\left[\Phi(z) + \bar{\Phi}(\bar{z})\right]^2}{2\Phi'(z)\,\bar{\Phi}\,'(\bar{z})}\; u(z)\,\bar{u}(\bar{z})\,, \tag{2.20}$$

depends on two chiral $(u,\,\Phi)$ and two antichiral $(\bar{u},\,\bar{\Phi})$ functions. The parameterization (2.11) defines a 'constraint surface' in the space of fundamental quadratic forms and it can be treated as a gauge fixing condition. In fact, a gauge fixing condition in (2.20) can be written in the form

$$\frac{u(z)}{\Phi'(z)} = a = \frac{\bar{u}(\bar{z})}{\bar{\Phi}'(\bar{z})}\,, \tag{2.21}$$

where a is a constant. Then, with $f(z) = a\Phi(z)$ and $\bar{f}(\bar{z}) = -a\bar{\Phi}(\bar{z})$ we obtain (2.11).

In order to find independent parameterizing variables of string surfaces $X(\tau,\sigma)$, it is important to analyze the remaining freedom of conformal transformations in (2.11). This analysis is done in appendix B. It shows that the freedom of conformal transformations in (2.11) is described by three parameters ϕ_0, $\bar{\phi}_0$ and c. The first two correspond to translations in the chiral and antichiral sectors. The transformations parameterized by c are f, $\bar{f}$ dependent. Their infinitesimal form is defined by

$$\zeta'(z) = 1 + \varepsilon\, c(f(z) - p)\,, \qquad\qquad \bar{\zeta}'(\bar{z}) = 1 + \varepsilon\, c\left(\bar{f}(\bar{z}) - p\right)\,. \tag{2.22}$$

The variable c could be included in the infinitesimal parameter ε, however, it is more convenient to keep this form. In appendix B we also show that the conditions (2.19) are invariant under these conformal transformations. We use the remaining conformal symmetry to reduce the number of parameterizing variables.

Let's consider Lorentz transformations of the basis (2.15). A boost in X_1-direction transforms the basis $(\mathbf{e}_+,\,\mathbf{e}_-,\,\mathbf{e})$ to $(P\,\mathbf{e}_+,\,P^{-1}\,\mathbf{e}_-,\,\mathbf{e})$, where $P > 0$ and $\theta = \log P$ is the boost parameter. The corresponding tangent vectors (2.12) become P dependent

$$B = f\,\mathbf{e} + P\,\mathbf{e}_+ + f^2\,P^{-1}\,\mathbf{e}_-\,, \qquad \bar{B} = \bar{f}\,\mathbf{e} + P\,\mathbf{e}_+ + \bar{f}^2\,P^{-1}\,\mathbf{e}_-\,. \tag{2.23}$$

Below we show that this equation defines the general form of the tangent vectors. Namely, further Lorentz transformations of the basis (2.15) correspond either to transformed parameterizing variables $(f(z),\, \bar{f}(\bar{z}), P)$, or to the remaining conformal freedom.

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We divided infinitesimal Lorentz transformations in three independent groups

$$\begin{aligned}
&1.\quad \mathbf{e}_+ \mapsto \mathbf{e}_+ + \varepsilon \mathbf{e}_+\,, && \mathbf{e}_- \mapsto \mathbf{e}_- - \varepsilon \mathbf{e}_-\,, && \mathbf{e} \mapsto \mathbf{e}\,; && (2.24)\\
&2.\quad \mathbf{e}_+ \mapsto \mathbf{e}_+ + \varepsilon \mathbf{e}\,, && \mathbf{e}_- \mapsto \mathbf{e}_-\,, && \mathbf{e} \mapsto \mathbf{e} + 2\varepsilon \mathbf{e}_-\,; && (2.25)\\
&3.\quad \mathbf{e}_+ \mapsto \mathbf{e}_+\,, && \mathbf{e}_- \mapsto \mathbf{e}_- + \varepsilon \mathbf{e} && \mathbf{e} \mapsto \mathbf{e} + 2\varepsilon \mathbf{e}_+\,. && (2.26)
\end{aligned}$$

Eq. (2.24) corresponds to a boost in X_1-direction, whereas (2.25) and (2.26) are two different compositions of a boost in X_2-direction and a rotation in (X_1, X_2)-plane.

The transformations (2.24) and (2.25) preserve the structure of the tangent vectors (2.23) with transformed $(f,\, \bar{f},\, P)$. From (2.24) and (2.23) one gets

$$f \mapsto f\,, \qquad \bar{f} \mapsto \bar{f}\,, \qquad P \mapsto P + \varepsilon P\,, \tag{2.27}$$

and similarly (2.25) leads to

$$f \mapsto f + \varepsilon P\,, \qquad \bar{f} \mapsto \bar{f} + \varepsilon P\,, \qquad P \mapsto P\,. \tag{2.28}$$

Eq. (2.27) is consistent with the definition of P, as a boost parameter in X_1-direction, and eq. (2.28) states that the Lorentz transformations (2.25) correspond to translations of the zero modes of $f(z)$ and $\bar{f}(\bar{z})$. This means that the parameters related to the transformations (2.24) and (2.25) can be neglected, since the corresponding freedom is encoded in dilatations of P and translations of f and $\bar{f}$.

The transformations (2.26) are of different type. They change the tangent vector B in (2.23) in the following way

$$B \mapsto B_\varepsilon = \left(f + \varepsilon\, f^2\, P^{-1}\right) \mathbf{e} + (P + 2\varepsilon\, f)\, \mathbf{e}_+ + f^2\, P^{-1}\, \mathbf{e}_-\,. \tag{2.29}$$

This destroys the structure of (2.23). In particular, the $\mathbf{e}_+$- components of the transformed tangent vector B_ε is not constant anymore. The same is valid for $\bar{B}_\varepsilon$.

Here we use the remaining conformal freedom (2.22). An infinitesimal transformation $z \mapsto z + \varepsilon\phi(z)$ corresponds to

$$B_\varepsilon(z) \mapsto \left(1 + \varepsilon\phi'(z)\right) \left[B_\varepsilon(z) + \varepsilon\, \phi(z) B'_\varepsilon(z)\right]\,, \tag{2.30}$$

and the coefficient of $\mathbf{e}_+$ becomes constant with

$$\phi'(z) = -2\, P^{-1}\, (f(z) - p)\,, \tag{2.31}$$

which is an allowed conformal transformation (2.22) with $c = -2\, P^{-1}$. The transformed constant coefficient of $\mathbf{e}_+$ is equal to $P + 2\varepsilon p$ and it is easy to check that the transformed coefficients of $\mathbf{e}_-$ and $\mathbf{e}$ are related as in (2.23).

Summarizing the discussion on Lorentz transformations of the basis (2.15), we conclude that eq. (2.23) indeed describes the general form of the tangent vectors.

From (2.23) follows that $\partial_\tau X_+ = P$ and $\partial_\sigma X_+ = 0$, where X_+ is the $\mathbf{e}_+$ component of $X(\tau, \sigma)$. These are the standard light cone gauge conditions in 3d bosonic string.

The above mentioned integration constants of the equations $\partial X = B$ and $\bar{\partial} X = \bar{B}$, together with the freedom in $(\phi_0,\, \bar{\phi}_0)$-translations, describe the coordinate zero modes of

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$X(\tau,\sigma)$ in the lightcone gauge. Thus, the parameterization of the first and the second fundamental forms by (2.11), after factorization of the remaining conformal symmetry, corresponds to the lightcone gauge.

Now we analyze the conformal factor of the metric tensor e^α, defined by (2.11). The function $\bar f - f$ is given as a sum of the non-zero Fourier modes

$$\bar f - f = \sum_{n\neq 0}\left[\bar a_n\, e^{-in(\tau-\sigma)} - a_n\, e^{-in(\tau+\sigma)}\right], \tag{2.32}$$

and its integration by σ around the unit circle vanishes. This means that $\bar f(\bar z) - f(z)$ has not a fixed sign. The points where this function vanishes correspond to the above mentioned degeneracy of the induced metric, i.e. $\partial_\tau X\cdot\partial_\tau X = 0 = \partial_\sigma X\cdot\partial_\sigma X$ and $\alpha\to-\infty$. Calculating the tangent vector $\partial_\sigma X = B-\bar B$, from (2.12) we find

$$\partial_\sigma X = (f-\bar f)\,\left[\,\mathbf{e} + (f+\bar f)\,\mathbf{e}_{\_}\,\right], \tag{2.33}$$

which vanishes at $\bar f(\bar z) - f(z) = 0$. Note that the normal vector (2.13) diverges at these points . Since this vector has the unit norm, it diverges in the lightlike direction.

The worldsheet scalar curvature, calculated in the conformal coordinates, is given by $R = -2e^{-\alpha}\,\bar\partial\partial\,\alpha$. In the parameterization (2.11), it takes the form

$$R = -\frac{8\,f'(z)\,\bar f'(\bar z)}{\left[\bar f(\bar z) - f(z)\right]^4}, \tag{2.34}$$

which is singular at $\bar f(\bar z) - f(z) = 0$. This singularity can not be removed by coordinate transformations. Hence, the lightcone gauge string surfaces in three dimensions are always singular.

In higher dimensions, the conformal factor of the induced metric tensor in the lightcone gauge is given by

$$e^\alpha = \frac{1}{2}\sum_a[\bar f_a(\bar z) - f_a(z)]^2, \tag{2.35}$$

where the summation index a corresponds to the transverse (to $\mathbf{e}_+$ and $\mathbf{e}_{\_}$) coordinates. Each $\bar f_a - f_a$ has the structure (2.32) and, therefore, they vanish at some points. But if these points for different a's do not coincide, α is globally regular.

The Fourier modes in (2.18) are canonical variables, which are used for the quantization of the lightcone bosonic string [19]. The key point for a consistent quantization is to check the commutation relations of the Poincare group generators. This calculation in an arbitrary dimension of spacetime is non-trivial only for the commutators of the Lorentz transformations $[M_{-\,a}, M_{-\,b}]$, where a and b are indices for the transverse coordinates. The Poincare symmetry requires vanishing of these commutator, which in 3 dimensions is trivially fulfilled, since there is only one transverse coordinate. So, there is no quantum anomaly in the Poincare algebra of the lightcone quantized 3d bosonic string. However, it appears that there is an additional class of string solutions in three dimensions, which is not covered by the lightcone gauge strings.

Before introducing the new class, we describe those properties of u and $\bar{u}$, which distinguish the classes. These functions have vanishing zero modes

$$\int_0^{2\pi} \mathrm{d}z\; u(z) = 0 = \int_0^{2\pi} \mathrm{d}\bar{z}\; \bar{u}(\bar{z})\,, \tag{2.36}$$

since $u = f'$, $\bar{u} = \bar{f}'$ and f, $\bar{f}$ are periodic. Note that if $u = 0$, $\bar{u} = 0$, the tangent vectors (2.23) become constants and the string surface degenerates to a massless particle trajectory. Neglecting this degenerated case, from (2.36) follows that $u(z)$ and $\bar{u}(\bar{z})$ change signs in the interval of periodicity. This property is obviously invariant under the conformal transformations (2.8).

Thus, the lightcone gauge describes the string surfaces with changing signs of $u(z)$ and $\bar{u}(\bar{z})$. In the next subsection we show that the class of string surfaces with fixed signs of $u(z)$ and $\bar{u}(\bar{z})$ is not empty, and this class describes oscillating circular and rotating spiky strings.

There is an additional class of $u(z)$, $\bar{u}(\bar{z})$, which have zeros, but do not change signs there. It is an 'intermidiate' class between the lightcone and spiky strings. The corresponding surfaces have different type of singularities, which 'move' in the lightcone directions around the (τ, σ)-cylinder. We do not consider this class in this paper.

2.3 Liouville gauge

Suppose $u(z)$ and $\bar{u}(\bar{z})$ have no zeros. Such functions can be transformed to constants $u(z) \mapsto \lambda$, $\bar{u}(\bar{z}) \mapsto \bar{\lambda}$ by the conformal transformation (2.9). The dynamical variables λ and $\bar{\lambda}$ have the same signs as u and $\bar{u}$, respectively, and their modules are given by the conformal invariants

$$2\pi\sqrt{|\lambda|} = \int_0^{2\pi} dz\,\sqrt{|u(z)|}\,, \qquad 2\pi\sqrt{|\bar{\lambda}|} = \int_0^{2\pi} d\bar{z}\,\sqrt{|\bar{u}(\bar{z})|}\,, \tag{2.37}$$

which easily follow from the monodromy properties of ζ and $\bar{\zeta}$.

The choice of constant $u(z)$ and $\bar{u}(\bar{z})$ fixes the conformal gauge freedom up to zero modes of ζ and $\bar{\zeta}$. We call this choice the Liouville gauge, since the corresponding consistency condition (2.7) reduces to the Liouville equation

$$\partial\bar{\partial}\alpha + \lambda\bar{\lambda}\, e^{-\alpha} = 0\,. \tag{2.38}$$

The general solution of this equation is given by

$$e^{-\alpha} = \frac{2}{|\lambda\bar{\lambda}|}\;\frac{F'(z)\,\bar{F}'(\bar{z})}{\big[\epsilon F(z) + \bar{\epsilon}\bar{F}(\bar{z})\big]^2}\,, \tag{2.39}$$

where F, $\bar{F}$ are monotonic functions $F' > 0$, $\bar{F}' > 0$ and $\epsilon = \text{sign}\,\lambda$, $\bar{\epsilon} = \text{sign}\,\bar{\lambda}$. For a symmetry reason, we treat all four possibilities of $(\epsilon,\,\bar{\epsilon})$ simultaneously, though the general solution (2.39) depends only on the sign of $\epsilon\bar{\epsilon}$.

The integration of the linear system (2.5) with α given by (2.39) and $u(z) = \lambda$, $\bar{u}(\bar{z}) = \bar{\lambda}$ can be done similarly to the lightcone gauge. Repeating the same steps as

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before (see appendix A) we obtain

$$B=\frac{\lambda}{F'(z)}\left[\ F(z)\ \mathbf{e}+\epsilon\,\mathbf{e}_{+}+\epsilon\,F^{2}(z)\,\mathbf{e}_{-}\right],$$
$$\bar{B}=\frac{\bar{\lambda}}{\bar{F}'(\bar{z})}\left[-\bar{F}(\bar{z})\,\mathbf{e}+\bar{\epsilon}\,\mathbf{e}_{+}+\bar{\epsilon}\,\bar{F}^{2}(\bar{z})\,\mathbf{e}_{-}\right], \tag{2.40}$$
$$N=\frac{\bar{\epsilon}\,\bar{F}(\bar{z})-\epsilon\,F(z)}{\epsilon\,F(z)+\bar{\epsilon}\,\bar{F}(\bar{z})}\,\mathbf{e}-\frac{2}{\epsilon\,F(z)+\bar{\epsilon}\,\bar{F}(\bar{z})}\,\mathbf{e}_{+}+\frac{2\,\bar{\epsilon}\,\epsilon\,\bar{F}(\bar{z})\,F(z)}{\epsilon\,F(z)+\bar{\epsilon}\,\bar{F}(\bar{z})}\,\mathbf{e}_{-}\,, \tag{2.41}$$

Here $(\mathbf{e},\mathbf{e}_{+},\mathbf{e}_{-})$ are again $\mathbb{R}^{1,2}$-valued integration constants with the same orthonormality conditions (2.14).

The space of Liouville fields (2.39) is invariant under the transformations

$$e^{-\alpha(z,\bar{z})}\mapsto\zeta'(z)\bar{\zeta}'(\bar{z})\ e^{-\alpha(\zeta(z),\bar{\zeta}(\bar{z}))}\,, \tag{2.42}$$

which corresponds to $F(z)\mapsto F(\zeta(z))$, $\bar{F}(\bar{z})\mapsto\bar{F}(\bar{\zeta}(\bar{z}))$. These are the conformal transformations in Liouville theory. In spite of similarity, there is an essential differences between the conformal transformations (2.8) and (2.42). Namely, the transformations (2.8) describe the freedom in choice of conformal coordinates and they do not change the string surface. Whereas, (2.42) acts on the Liouville fields and it changes the date of the linear system (2.5), which is not a worldsheet reparameterization. Note also that the conformal weight of $e^{-\alpha}$ is equal to 1 by (2.42) and -1 by (2.8). To avoid misunderstanding with these two conformal transformations, we use for (2.42) and the related maps in Liouville theory the name Virasoro transformations.

Let us discuss the regularity issue of closed string worldsheets in the Liouville gauge, related to peculiarities of the Liouville field $\alpha(\tau,\sigma)$. It is well known that a globally regular Liouville field on a cylindrical spacetime exist only for $\epsilon\bar{\epsilon}=-1$ and it belongs to the hyperbolic monodromy [20]. The parameterizing functions of this monodromy class satisfy the conditions $F(z+2\pi)=e^{P}\,F(z)$ and $\bar{F}(\bar{z}+2\pi)=e^{P}\,\bar{F}(\bar{z})$, with $P>0$.

However, these conditions do not correspond to periodic tangent vectors (2.40). It means that the linear system (2.5) does not provide a closed string configuration in the Liouville gauge, if the Liouville field on the cylinder is regular.

There is a class of singular Liouville fields with a regular stress tensor of the theory and some other remarkable properties [16, 21, 22]. Singularities of these fields correspond to zeros of the exponent e^{α}. The equation $e^{\alpha(\tau,\sigma)}=0$ is equivalent to $\epsilon F(z)+\bar{\epsilon}\bar{F}(\bar{z})=0$ and it defines non-intersecting, smooth lines on the $(\tau,\,\sigma)$-manifold. It appears that this type of singular Liouville fields on the cylinder can match the periodicity conditions of closed string dynamics.

The authors of ref. [16] gave a complete classification of periodic Liouville fields by the coadjoint orbits of the Virasoro algebra. This classification is based on the analysis of the Schrödinger (Hill) equation with a periodic potential given by the stress tensor of Liouville theory. We use these results here to select an appropriate class of Liouville fields. In appendix C we give a list of relations in Liouville theory which are helpful for understanding of the technical details below.

The solutions of Hill equations (C.4) in the chiral and the antichiral sectors are denoted by $\psi(z)$, $\chi(z)$ and $\bar\psi(\bar z)$, $\bar\chi(\bar z)$, respectively. They are normalized by the unit Wronskians (C.5). The parameterization of the functions $F(z)$ and $\bar F(\bar z)$ in terms of these solutions (C.7) define the following form of the tangent vectors (2.40)

$$B = \lambda\epsilon\left[\psi(z)\;\chi(z)\;\mathbf{e} + \psi^2(z)\,\mathbf{e}_+ + \chi^2(z)\,\mathbf{e}_-\right] , \tag{2.43}$$
$$\bar B = \bar\lambda\bar\epsilon\left[-\bar\psi(\bar z)\,\bar\chi(\bar z)\,\mathbf{e} + \bar\chi^2(\bar z)\,\mathbf{e}_+ + \bar\psi^2(\bar z)\,\mathbf{e}_-\right] . \tag{2.44}$$

The periodicity conditions $B(z+2\pi) = B(z)$, $\bar B(\bar z + 2\pi) = \bar B(\bar z)$ require

$$\psi(z+2\pi) = \pm\psi(z)\,, \qquad \chi(z+2\pi) = \pm\chi(z)\,, \tag{2.45}$$
$$\bar\psi(\bar z+2\pi) = \pm\bar\psi(\bar z)\,, \qquad \bar\chi(\bar z+2\pi) = \pm\bar\chi(\bar z)\,, \tag{2.46}$$

which corresponds to the monodromy matrix $\pm I$. In the classification of the coadjoint orbits this class is denoted by $E_\pm$. Its typical representatives are

$$\psi_k(z) = \sqrt{\frac{2}{k}}\,\cos\left(\frac{kz}{2}\right) , \qquad \chi_k(z) = \epsilon\sqrt{\frac{2}{k}}\,\sin\left(\frac{kz}{2}\right) , \tag{2.47}$$
$$\bar\psi_{\bar k}(\bar z) = \sqrt{\frac{2}{\bar k}}\,\cos\left(\frac{\bar k\bar z}{2}\right) , \qquad \bar\chi_{\bar k}(\bar z) = -\bar\epsilon\sqrt{\frac{2}{\bar k}}\,\sin\left(\frac{\bar k\bar z}{2}\right) ,$$

where k and $\bar k$ are positive integers. They count the number of zeros of these functions in the interval $[0, 2\pi)$. The coefficients in front of sin and cos-functions correspond to the normalization of Wronskians (C.5). The corresponding Liouville field configurations are associated with vacuum solutions, since the stress tensor for these fields is constant

$$T(z) = -\frac{k^2}{4}\,, \qquad \bar T(\bar z) = -\frac{\bar k^2}{4}\,. \tag{2.48}$$

The general representatives of $E_\pm$, are obtained by the Virasoro transformations of the functions (2.47) with the conformal weight $-\frac{1}{2}$

$$\psi_k(z) \mapsto \left(\zeta'(z)\right)^{-\frac{1}{2}}\,\psi_k(\zeta(z))\,, \qquad \chi_k(z) \mapsto \left(\zeta'(z)\right)^{-\frac{1}{2}}\,\chi_k(\zeta(z))\,, \tag{2.49}$$
$$\bar\psi_k(\bar z) \mapsto \left(\bar\zeta'(\bar z)\right)^{-\frac{1}{2}}\,\bar\psi_k(\bar\zeta(\bar z))\,, \qquad \bar\chi_k(\bar z) \mapsto \left(\bar\zeta'(\bar z)\right)^{-\frac{1}{2}}\,\bar\chi_k(\bar\zeta(\bar z))\,.$$

Thus, the acceptable class of Liouville fields is associated with the Virasoro group orbits of the vacuum configurations.

Let us consider the string configurations related to (2.47) in more detail. In this case the tangent vectors (2.43) read

$$B(z) = \Lambda\,\left[(1+\cos(nz))\,\mathbf{e}_+ + (1-\cos(nz))\,\mathbf{e}_- + \sin(nz)\;\mathbf{e}\right] , \tag{2.50}$$
$$\bar B(\bar z) = \bar\Lambda\,\left[(1-\cos(\bar n\bar z))\,\mathbf{e}_+ + (1+\cos(\bar n\bar z))\,\mathbf{e}_- + \sin(\bar n\bar z)\,\mathbf{e}\right] ,$$

where we have introduced the notations

$$\Lambda = \frac{|\lambda|}{k}\,, \qquad \bar\Lambda = \frac{|\bar\lambda|}{\bar k}\,, \qquad n = \epsilon k\,, \qquad \bar n = \bar\epsilon\bar k\,. \tag{2.51}$$

Note that Λ and $\bar{\Lambda}$ are positive and n, $\bar{n}$ are nonzero integers. It is easy to see that the periodicity of $X(\tau,\sigma)$ requires $\Lambda = \bar{\Lambda}$.

To proceed, we choose the constant basis vectors as in (2.15)

$$\mathbf{e} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{e}_{+} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_{-} = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \tag{2.52}$$

and rewrite eq. (2.50) in the form

$$B = \Lambda \begin{pmatrix} 1 \\ \cos(nz) \\ \sin(nz) \end{pmatrix}, \qquad \bar{B} = \Lambda \begin{pmatrix} 1 \\ -\cos(\bar{n}\bar{z}) \\ \sin(\bar{n}\bar{z}) \end{pmatrix}. \tag{2.53}$$

The integration of these tangent vectors, up to translations, yields the string surface

$$X(\tau,\sigma) = \frac{\Lambda}{n\bar{n}} \begin{pmatrix} 2n\bar{n}\tau \\ \bar{n}\sin(n\tau + n\sigma) \ - \ n\sin(\bar{n}\tau - \bar{n}\sigma) \\ -\bar{n}\cos(n\tau + n\sigma) \ - \ n\cos(\bar{n}\tau - \bar{n}\sigma) \end{pmatrix}. \tag{2.54}$$

Thus, the time component is proportional to τ

$$X^0 = 2\Lambda\tau\,, \tag{2.55}$$

and the spatial part is described by the function

$$Z(\tau,\sigma) = \frac{\Lambda}{in}\, e^{i(n\tau + n\sigma)} + \frac{\Lambda}{i\bar{n}}\, e^{-i(\bar{n}\tau - \bar{n}\sigma)}\,, \tag{2.56}$$

where $Z = X_1 + iX_2$ is the complex coordinate on the (X_1, X_2)-plane.

The conformal factor (2.3) of the induced metric is given by

$$e^{\alpha} = \Lambda^2 \left[1 + \cos(nz + \bar{n}\bar{z})\right]\,, \tag{2.57}$$

and this function vanishes at

$$(n + \bar{n})\tau + (n - \bar{n})\sigma = (2m+1)\pi\,, \qquad (m \in \mathbb{Z})\,. \tag{2.58}$$

The zeros of e^{α} correspond to $\partial_\tau X \cdot \partial_\tau X = 0 = \partial_\sigma X \cdot \partial_\sigma X$. From eq. (2.53) follows that the vector $\partial_\tau X = B + \bar{B}$ is indeed lightlike at the singular points (2.58) and $\partial_\sigma X = B - \bar{B}$ vanishes there. The conditions $\partial_\sigma X = 0$ and the lightlikeness of $\partial_\tau X$ are the boundary conditions for an open string. Therefore, the singular points on the worldsheet look like the end points of an open string and they have a spiky character.

According to (2.58), the number of spikes for a fixed τ is equal to $|n - \bar{n}|$. The case $\bar{n} = n$ is special and we consider it separately.

The string worldsheet (2.54) for $\bar{n} = n$ reduces to

$$X(\tau,\sigma) = \frac{2\Lambda}{n} \begin{pmatrix} n\tau \\ \cos(n\tau)\ \sin(n\sigma) \\ -\cos(n\tau)\ \cos(n\sigma) \end{pmatrix}. \tag{2.59}$$

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The corresponding string configuration at a fixed τ is a circle on the $(X_1,\, X_2)$-plane with the center at the origin. The radius of the circle oscillates in time. At $\tau = (m+\frac{1}{2})\frac{\pi}{n}$, $m \in Z$, the circle shrinks to the origin. Thus, the case $\bar{n} = n$ describes oscillating circular strings without spikes.

Now we consider the general case with an arbitrary n and $\bar{n}$, except $\bar{n} = n$.

From eq. (2.56) follows the relation

$$Z(\tau,\sigma) = e^{i\omega\tau}\, Z(\sigma+\omega_0\tau)\,, \tag{2.60}$$

where

$$\omega = -\frac{2n\bar{n}}{n-\bar{n}}\,, \qquad \omega_0 = \frac{n+\bar{n}}{n-\bar{n}}\,, \tag{2.61}$$

and $Z(\sigma) = Z(0,\sigma)$ corresponds to the string configuration at $\tau = 0$. The shift of the argument σ by $\omega_0\tau$ does not change the string shape. Therefore, eq. (2.60) describes a rotating string around the origin with the frequency ω. The sign of ω defines the direction of the rotation. Thus, the initial shape is preserved in dynamics.

Using again (2.54), the shape of the string configuration at $\tau = 0$ can be written as

$$Z(\sigma) = \frac{\Lambda}{in}\, e^{in\sigma} + \frac{\Lambda}{i\bar{n}}\, e^{i\bar{n}\sigma}\,. \tag{2.62}$$

This curve, given as a composition of two 'rotations', is known as a epicycloid if n and $\bar{n}$ have the same sign and a hypocycloid if their signs are opposite. The rotation parameter is σ. The integers n and $\bar{n}$ define the rotation frequencies and the rotation radiuses are given by their inverse numbers, up to the scale factor Λ.

In appendix D we give several plots of $Z(\sigma)$ for different values of n and $\bar{n}$. Below we list some properties of the curves (2.62), which help to understand the structure of these plots and also to visualize the general case.

1. The spikes at $\tau = 0$ correspond to the points

$$\sigma_m = \frac{2m+1}{|n-\bar{n}|}\,\pi\,, \qquad m = 0,\, 1, \ldots,\, |n-\bar{n}|-1\,, \tag{2.63}$$

   and they are located on the circle with the center at the origin and the radius

$$|Z(\sigma_m)| = \frac{\Lambda\,|n-\bar{n}|}{|n\bar{n}|}\,. \tag{2.64}$$

2. The following inequalities hold for $\sigma \in (\sigma_m, \sigma_{m+1})$

$$\begin{aligned} &\frac{|n+\bar{n}|}{|n\bar{n}|} \le \frac{|Z(\sigma)|}{\Lambda} < \frac{|n-\bar{n}|}{|n\bar{n}|}\,, \qquad \text{for} \quad n\bar{n} < 0\,; \\ &\frac{|n-\bar{n}|}{|n\bar{n}|} < \frac{|Z(\sigma)|}{\Lambda} \le \frac{|n+\bar{n}|}{|n\bar{n}|}\,, \qquad \text{for} \quad n\bar{n} > 0\,. \end{aligned} \tag{2.65}$$

3. The spike corresponding to σ_m has the direction of the radius vector at $Z(\sigma_m)$ if $n\bar{n} < 0$; and the direction of the spike is opposite to the radius vector, if $n\bar{n} > 0$.

4. The curvature of $Z(\sigma)$, $\sigma \in (\sigma_m, \sigma_{m+1})$, with respect to the origin is positive, if $n\bar{n} > 0$; it is zero, if $n + \bar{n} = 0$; and it is negative, if $n\bar{n} < 0$ and $n + \bar{n} \neq 0$.

5. When σ changes from σ_m to σ_{m+1}, the polar angle of $Z(\sigma)$ rotates on

$$\Delta\phi = 2\pi \, \frac{\min(|n|, |\bar{n}|)}{|n - \bar{n}|} \, . \tag{2.66}$$

According to (2.64) and (2.65), the spikes are the farthest points from the origin, if $n\bar{n} < 0$, and they are the nearest ones, if $n\bar{n} > 0$. These two properties easily follow from eq. (2.62). The proof of other properties and some additional information about the spiky strings is given in appendix E. The derivation of eq. (2.66) there does not apply to the case $n + \bar{n} = 0$ (see eq. (E.9)). We consider this case here separately.

If $n + \bar{n} = 0$, eq. (2.54) provides the surface

$$X(\tau, \sigma) = \frac{2\Lambda}{n} \begin{pmatrix} n\tau \\ \cos(n\tau)\,\sin(n\sigma) \\ \sin(n\tau)\,\sin(n\sigma) \end{pmatrix} , \tag{2.67}$$

which describes a folded (n-times) rotating string. The spikes are at $\sigma_m = (m + \frac{1}{2})\frac{\pi}{|n|}$, $m = 0, 1, \ldots, 2|n| - 1$. They correspond to one end of the folded string for even m, and to another end for odd m. Thus, one gets $\Delta\phi = \pi$, which is consistent with (2.66).

Eq. (2.66) indicates that the curve $Z(\sigma)$ is non-intersecting if $\min(|n|, |\bar{n}|) = 1$ and $|n| \neq |\bar{n}|$. For example, if $\bar{n} = -1$ and $n > 1$, one gets the spiky strings of [12]

$$X(\tau, \sigma) = \frac{\Lambda}{n} \begin{pmatrix} 2n\tau \\ \sin(n\tau + n\sigma) - n\sin(\tau - \sigma) \\ -\cos(n\tau + n\sigma) + n\cos(\tau - \sigma) \end{pmatrix} . \tag{2.68}$$

In general, hypocycloids and epicycloids are intersecting (or folded) curves.

The properties 1 - 5 help to visualize these curves and draw them qualitatively. The properties 1 and 3 provide the positions of spikes and their directions, respectively. Other properties (2, 4 and 5) define how the spikes are connected by smooth curves. The form of these curves and the direction of the spikes essentially depend on the sign of $n\bar{n}$, as one can see in appendix D.

Since these curves correspond to a composition of two rotations, they arise in description of simple mechanical systems. Their properties were investigated long time ago and one can find them in the literature. Here we present them for completeness.

The string solutions given as rotating hypocycloids and epicycloids first were obtained in [23] as a model of hadrons. Later these solutions were rediscovered in cosmic strings and in AdS/CFT correspondence by different authors. A list of references and interesting comments one can find in a recent letter paper [24].

The general solution in the Liouville gauge is obtained by the Poincare and Virasoro transformations of the vacuum solutions (2.54). The Lorentz transformation in the Poincare group correspond to the freedom in choice of the basis $(\mathbf{e}, \mathbf{e}_+, \mathbf{e}_-)$ and the translations are related to the integration constants, which we have neglected in (2.54). The Virasoro transformations (2.49) map the tangent vectors (2.53) to

$$B(z) = \frac{\Lambda}{\zeta'(z)} \begin{pmatrix} 1 \\ \cos(n\zeta(z)) \\ \sin(n\zeta(z)) \end{pmatrix}, \qquad \bar{B}(\bar{z}) = \frac{\Lambda}{\bar{\zeta}'(\bar{z})} \begin{pmatrix} 1 \\ -\cos(\bar{n}\bar{\zeta}(\bar{z})) \\ \sin(\bar{n}\bar{\zeta}(\bar{z})) \end{pmatrix}. \tag{2.69}$$

To integrate these vectors to a periodic $X(\tau, \sigma)$, their zero modes have to be equal

$$\int_0^{2\pi} dz\, B(z) = \int_0^{2\pi} d\bar{z}\, \bar{B}(\bar{z}). \tag{2.70}$$

This equation relates ζ and $\bar{\zeta}$ by three (one for each vector component) conditions.

The induced metric now is degenerated at $n\zeta(z) + \bar{n}\bar{\zeta}(\bar{z}) = (2m+1)\pi \quad (m \in Z)$ and the tangent vector $\zeta'(z)\partial X - \bar{\zeta}'(\bar{z})\bar{\partial} X$ vanishes there.

Let us consider the following infinitesimal transformation

$$\zeta(z) = z + \varepsilon_0 + \frac{\varepsilon_1}{n}\sin(nz) + \frac{\varepsilon_2}{n}\cos(nz). \tag{2.71}$$

It is easy to check that the corresponding $B(z)$ in (2.69) is a Lorentz transformed vacuum vector from (2.53). In particular, ε_0 becomes an infinitesimal rotation angle in (X_1, X_2) plane, whereas ε_1 and ε_2 are infinitesimal boost parameters in X_1 and X_2 directions, respectively. Thus, the Virasoro transformations of the parameterizing Liouville field contain the Lorentz transformations of the target space as a subgroup.

As it was mentioned in the previous subsection, the lightcone gauge does not cover the string solutions of the Liouville gauge. Therefore, the complete quantum picture of $\mathbb{R}^{1,2}$ strings requires quantization of singular Liouville fields, which is an open and a challenging problem in its own right.

Finally, concluding this section, we discuss a possible generalization of the parameterization (2.11) and the Liouville gauge $u(z) = \lambda$, $\bar{u}(\bar{z}) = \bar{\lambda}$ to higher dimensional Minkowski spaces.

The Pohlmeyer scheme in $\mathbb{R}^{1,n}$ is described in appendix F. If $n > 2$, there is an additional gauge symmetry related to the freedom in choice of a basis in the normal space to the string surface. We fix this gauge and provide a parameterization of the worldsheet variables similarly to (2.11) (see eq. (F.8)), which leads to the lightcone gauge strings with the induced metric (2.35).

In $\mathbb{R}^{1,n}$ one has $n-1$ second quadratic forms given by u_a and $\bar{u}_a$ $(a = 2, \ldots, n)$. Though the functions u_a are not chiral for $n > 2$, their scalar combination satisfies the chirality condition $\bar{\partial}(u_a\, u_a) = 0$. If $u_a\, u_a \neq 0$, this scalar function can be transformed to a constant $(u_a u_a = \lambda^2)$ by a conformal transformation. Similarly, one can put $\bar{u}_a\, \bar{u}_a = \bar{\lambda}^2$. These conditions can be treated as a generalization of the Liouville gauge to higher dimensions. String solutions with these gauge fixing conditions were constructed in [18], and they can be written in a form similar to (2.40)

$$\begin{aligned} \partial X &= \frac{\lambda}{F'(z)} \left[\, F_a(z)\ \mathbf{e}_a + \mathbf{e}_+ + F^2(z)\, \mathbf{e}_- \right], \\ \bar{\partial} X &= \frac{\bar{\lambda}}{\bar{F}'(\bar{z})} \left[-\bar{F}_a(\bar{z})\, \mathbf{e}_a + \mathbf{e}_+ + \bar{F}^2(\bar{z})\, \mathbf{e}_- \right]. \end{aligned} \tag{2.72}$$

Here $\mathbf{e}_+$, $\mathbf{e}_-$, $\mathbf{e}_a$ is a generalization of the basis (2.15) to $\mathbb{R}^{1,n}$, and we have used the notations $F^2 = F_a\, F_a$, $F' = \sqrt{F'_a\, F'_a}$, $\bar{F}^2 = \bar{F}_a\, \bar{F}_a$, $\bar{F}' = \sqrt{\bar{F}'_a\, \bar{F}'_a}$.

To get a periodic tangent vector ∂X, one can take periodic F_a, with $F'_a\, F'_a \neq 0$. Though this construction fails for $n = 2$, for $n > 2$ it provides the main class of solutions, which are conformally equivalent to the lightcone gauge strings (F.10). As it was mentioned in the previous subsections, the induced metric tensor on these string surfaces is mostly non degenerated. The spiky string surfaces constructed in $\mathbb{R}^{1,2}$ naturally exist in higher dimensional spaces, where they become exotic solutions with a degenerated induced metric. They correspond to $F_a = 0$ for $a \geq 3$. Extending the class of F_a's in (2.72), one can look for a new type of spiky strings in higher dimensions.

Another interesting issue is the analysis of integrability of the consistency conditions (F.4)–(F.6) in the gauge $u_a u_a = \lambda^2$ and $\bar{u}_a \bar{u}_a = \bar{\lambda}^2$. Here one expects a Liouville (WZW) type integrability, due to the chiral structure of the solutions (2.72). As it was shown in [18], the case $n = 3$ reduces to the complex Liouville equation, which is integrable in a same way as (2.38). The analysis of higher dimensional cases is more complicated due to the nonabelian character of the $SO(n-1)$ gauge group encoded in eqs. (F.4)–(F.6).

Parameterizing the gauge potentials A_{ab}, $\bar{A}_{ab}$ by $SO(n-1)$ group valued fields

$$A = \partial h\, h^{-1}\,, \qquad \bar{A} = \bar{\partial}\bar{h}\, \bar{h}^{-1}\,, \tag{2.73}$$

and introducing gauge invariant variables

$$v = \bar{h}^{-1}\, u\,, \qquad \bar{v} = h^{-1}\, \bar{u}\,, \qquad g = \bar{h}^{-1}\, h\,, \tag{2.74}$$

one gets the chirality conditions $\bar{\partial} v = 0$, $\partial \bar{v} = 0$ and the dynamics for g reduces to a gauged WZW theory interacting with the α-field through an integrable potential. This type of reduction to gauged WZW theories is well known in AdS spaces [3]. The difference in $\mathbb{R}^{1,n}$ case is a simple chiral structure leading to Toda type integrability.

3 Closed strings in $\mathbf{SL(2,\mathbb{R})}$

We start this section with a standard approach to string dynamics in AdS spaces. Then we pass from AdS_3 to $\mathrm{SL}(2,\mathbb{R})$ group valued variables and introduce the chiral structure of WZW theory there. This enables us to formulate the Pohlmeyer type scheme in the same manner as for $\mathbb{R}^{1,2}$.

3.1 String dynamics in AdS_3

AdS_3 is realized as the hyperbola

$$Y \cdot Y + 1 = 0 \tag{3.1}$$

embedded in $\mathbb{R}^{2,2}$. $Y \equiv (Y^{\tilde{0}}, Y^0; Y^1, X^2)$ denotes a point in $\mathbb{R}^{2,2}$ and the scalar product $Y \cdot Y = Y^M Y^N\, G_{MN}$ is defined by the metric tensor $G_{MN} = \mathrm{diag}(-1,-1;1,1)$.

The choice of conformal coordinates on a timelike string worldsheet $Y(\tau,\sigma)$ assumes

$$\partial Y \cdot \partial Y = 0 = \bar{\partial} Y \cdot \bar{\partial} Y\,, \tag{3.2}$$

and the nonzero element of the induced metric tensor is parameterized as in (2.3)

$$\partial Y \cdot \bar{\partial} Y = -e^{\alpha}\,. \tag{3.3}$$

String dynamics in the conformal gauge is described by the Lagrangian

$$\mathcal{L} = \partial Y \cdot \bar{\partial} Y + \Lambda(Y \cdot Y + 1)\,, \tag{3.4}$$

where Λ is a Lagrange multiplier. Its elimination from the equation of motion yields

$$\bar{\partial}\partial Y + e^{\alpha}\,Y = 0\,. \tag{3.5}$$

The Pohlmeyer scheme for this system leads to the sinh-Gordon equation for the α field [5, 6]. Though the system is formally integrable, one can write in an explicit form only the string solutions corresponding to the sinh-Gordon solitons [8] (see also [25]). To improve the integrability of the system by the structure of WZW theory [26], we use the isometry between AdS_3 and the $\mathrm{SL}(2,\mathbb{R})$ group manifold.

3.2 Map to $\mathrm{SL}(2,\mathbb{R})$ and WZW theory

First we describe the isometry between the $sl(2,\mathbb{R})$ algebra and $\mathbb{R}^{1,2}$. Let us introduce the basis in $sl(2,\mathbb{R})$

$$t_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,, \qquad t_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \qquad t_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{3.6}$$

These three matrices (t_μ, $\mu = 0, 1, 2$) satisfy the relations

$$t_\mu\, t_\nu = \eta_{\mu\nu}\, I + \epsilon_{\mu\nu}{}^{\rho}\, t_\rho, \tag{3.7}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1)$ form the metric tensor of 3d Minkowski space, I denotes the unit matrix and $\epsilon_{\mu\nu\rho}$ is the Levi-Civita tensor with $\epsilon_{012} = 1$ (see (2.16)). The expansion of $a \in sl(2,\mathbb{R})$ in the basis (3.6), $a = a^\mu\, t_\mu$, provides a map $a \mapsto a^\mu$ from $sl(2,\mathbb{R})$ to $\mathbb{R}^{1,2}$. The inner product in $sl(2,\mathbb{R})$, introduced by the normalized trace $\langle a\, b\rangle = \frac{1}{2}\,\mathrm{tr}(a\, b)$, leads to $\langle t_\mu\, t_\nu\rangle = \eta_{\mu\nu}$ and makes this map isometric.

A helpful remark is in order here. The transformations of the adjoint representation $a \mapsto g\, a\, g^{-1}$ ($g \in \mathrm{SL}(2,\mathbb{R})$) preserve the inner product in $sl(2,\mathbb{R})$. Therefore, the matrixes $\Lambda^\mu{}_\nu = \langle\, t^\mu\, g\, t_\nu\, g^{-1}\,\rangle$ define the Lorentz transformations of $\mathbb{R}^{1,2}$. Since $\mathrm{SL}(2,\mathbb{R})$ is connected, $\langle\, t^\mu\, g\, t_\nu\, g^{-1}\,\rangle \in SO_{\uparrow}(1,2)$ and $\langle\, t^0\, g\, t_0\, g^{-1}\,\rangle \geq 1$.

Now we consider the map from $Y \in AdS_3$ to $g \in \mathrm{SL}(2,\mathbb{R})$

$$g = \begin{pmatrix} Y^{\tilde{0}} + Y^2 & Y^1 + Y^0 \\ Y^1 - Y^0 & Y^{\tilde{0}} - Y^2 \end{pmatrix}\,, \tag{3.8}$$

which provides the equivalence between eq. (3.1) and the condition $\det g = 1$. Eq. (3.8) can be written in the form $g = Y^{\tilde{0}}\, I + Y^\mu\, t_\mu$. Therefore, the inverse map is given by

$$Y^{\tilde{0}} = \langle\, g\,\rangle\,, \qquad Y^\mu = \langle\, t^\mu\, g\,\rangle\,. \tag{3.9}$$

Note that $g^{-1} = Y^{\tilde{0}} I - Y^{\mu} t_{\mu}$. Using these compact forms of g and g^{-1} in terms of t_{μ} matrices, and the algebraic relations (3.7), one easily checks that

$$\left\langle g^{-1} dg \; g^{-1} dg \right\rangle = dY \cdot dY \,. \tag{3.10}$$

Due to this isometry, the conformal gauge conditions (3.2) are equivalent to

$$\left\langle g^{-1} \partial g \; g^{-1} \partial g \right\rangle = 0 = \left\langle g^{-1} \bar{\partial} g \; g^{-1} \bar{\partial} g \right\rangle \,, \tag{3.11}$$

and the parameterization of the nonzero component of the worldsheet metric by (3.3) can be written as

$$\left\langle g^{-1} \partial g \, g^{-1} \bar{\partial} g \right\rangle = -e^{\alpha} \,. \tag{3.12}$$

The AdS_3 string dynamics in the conformal gauge is described by the action

$$S_0 = \int dz \, d\bar{z} \, \langle g^{-1} \partial g \; g^{-1} \bar{\partial} g \rangle \,. \tag{3.13}$$

Its variation leads to the equation of motion

$$\partial \left(g^{-1} \bar{\partial} g \right) + \bar{\partial} \left(g^{-1} \partial g \right) = 0 \,, \tag{3.14}$$

which corresponds to (3.5), together with (3.1) and (3.3).

To get the equations of WZW theory [26]

$$\bar{\partial} \left(\partial g \, g^{-1} \right) = 0 = \partial \left(g^{-1} \bar{\partial} g \right) \,, \tag{3.15}$$

one has to add to the action (3.13) the WZ-term, which is a volume integral of the 3-form $H = \frac{1}{3} \langle g^{-1} \mathrm{d}g \wedge g^{-1} \mathrm{d}g \wedge g^{-1} \mathrm{d}g \rangle$. This form on $\mathrm{SL}(2, \mathbb{R})$ is exact $H = \mathrm{d}F$, with

$$F = \frac{\left\langle t_0 \; g^{-1} \; \mathrm{d}g \right\rangle \wedge \left\langle t_0 \; \mathrm{d}g \; g^{-1} \right\rangle}{1 + \langle t^0 \; g \; t_0 \; g^{-1} \rangle} \,. \tag{3.16}$$

Note that this 2-form is globally well defined due to the remark above. Then, with Stokes' theorem, the action of the $\mathrm{SL}(2, \mathbb{R})$ WZW theory is given by a surface integral from the Lagrangian [27]

$$\mathcal{L}_{\mathrm{WZW}} = \left\langle g^{-1} \partial g \; g^{-1} \bar{\partial} g \right\rangle + \frac{\left\langle t_0 \; g^{-1} \partial g \; \right\rangle \left\langle t_0 \; \bar{\partial} g \; g^{-1} \right\rangle - \left\langle t_0 \; g^{-1} \bar{\partial} g \right\rangle \left\langle t_0 \; \partial g \; g^{-1} \right\rangle}{1 + \langle t^0 \; g \; t_0 \; g^{-1} \rangle} \,. \tag{3.17}$$

The Euler-Lagrange equations obtained from (3.17) reproduce the equations of WZW theory (3.15). Adding to these equations the conformal gauge conditions (3.11), one gets a system called the $\mathrm{SL}(2, \mathbb{R})$ string [17].

In the next subsection we investigate this system by the Pohlmeyer scheme.

3.3 Pohlmeyer scheme for $\mathrm{SL}(2, \mathbb{R})$ string

Before starting the Pohlmeyer scheme note that a reparameterization invariant description of the system, yielding both the equation of motion (3.15) and the constraints (3.15), is given by the action

$$\begin{aligned} S = -\frac{1}{2} \int d^2 \xi \Bigg[& \sqrt{|h|} \; h^{ab} \left\langle g^{-1} \partial_a g \; g^{-1} \partial_b g \right\rangle \\ & + \epsilon^{ab} \, \frac{\left\langle t_0 \; g^{-1} \partial_a g \; \right\rangle \left\langle t_0 \; \partial_b g \; g^{-1} \right\rangle - \left\langle t_0 \; g^{-1} \partial_b g \right\rangle \left\langle t_0 \; \partial_a g \; g^{-1} \right\rangle}{1 + \langle t^0 \; g \; t_0 \; g^{-1} \rangle} \Bigg] \,. \end{aligned} \tag{3.18}$$

Here ξ^a $(a = 0,1)$ are worldsheet coordinates, h is the determinant of the worldsheet metric tensor h_{ab}, h^{ab} is its inverse and ϵ^{ab} is the 2d Levi-Civita tensor with $\epsilon^{01} = 1$. In the conformal gauge $h_{ab} \sim \mathrm{diag}\,(-1,1)$, we indeed obtain eqs. (3.11) and (3.15), with $\xi^0 = \tau$ and $\xi^1 = \sigma$.

Let us consider the Kac-Moody currents

$$J = \partial g\, g^{-1}\,, \qquad\qquad \bar{J} = g^{-1}\,\bar{\partial} g\,, \tag{3.19}$$

which, according to (3.15), satisfy the chirality conditions

$$\bar{\partial} J = 0 = \partial \bar{J}\,. \tag{3.20}$$

The parameterization of the induced metric (3.12) in terms of these currents reads

$$\left\langle\, J\, g\, \bar{J}\, g^{-1}\,\right\rangle = -e^{\alpha}\,, \tag{3.21}$$

and the conformal gauge conditions (3.11) are

$$\langle\, J\, J\rangle = 0 = \left\langle\, \bar{J}\, \bar{J}\right\rangle\,. \tag{3.22}$$

Due to the isometry between $sl(2,\mathbb{R})$ and $\mathbb{R}^{1,2}$, $J(z)$ and $\bar{J}(\bar{z})$ are associated with lightlike vectors as $B(z) = \partial X$ and $\bar{B}(\bar{z}) = \bar{\partial} X$ in $\mathbb{R}^{1,2}$.

Similarly to the $\mathbb{R}^{1,2}$ case, we consider the inner product of the Kac-Moody currents $\langle\, J\, \bar{J}\,\rangle$. Taking into account that J and $\bar{J}$ are lightlike and $g\, \bar{J}\, g^{-1}$ corresponds to a proper Lorentz transformation of $\bar{J}$, one finds that $\langle\, J\, \bar{J}\,\rangle$ and $\langle\, J\, g\, \bar{J}\, g^{-1}\,\rangle$ have the same sign. Therefore, we can use the parameterization

$$\left\langle\, J\, \bar{J}\,\right\rangle = -e^{\beta}\,. \tag{3.23}$$

Thus, the data for the Kac-Moody currents $(J,\, \bar{J})$ and the tangent vectors $(B,\, \bar{B})$ are similar. Using then the isometry between $sl(2,\mathbb{R})$ and $\mathbb{R}^{1,2}$, one gets the same linear system for a moving basis as (2.5)

$$\begin{aligned} \partial J &= \partial\beta\, J + v\, K\,, & \qquad\qquad \bar{\partial} J &= 0\,,\\ \partial \bar{J} &= 0\,, & \bar{\partial}\bar{J} &= \bar{\partial}\beta\, \bar{J} + \bar{v}\, K\,,\\ \partial K &= e^{-\beta}\, v\, \bar{J}\,, & \bar{\partial} K &= e^{-\beta}\, \bar{v}\, J\,. \end{aligned} \tag{3.24}$$

Here K is a $sl(2,\mathbb{R})$ valued unit vector, orthogonal to J and $\bar{J}$

$$\langle\, K\, K\,\rangle = 1\,, \qquad \langle\, K\, J\,\rangle = 0 = \left\langle\, K\, \bar{J}\,\right\rangle\,, \tag{3.25}$$

and v, $\bar{v}$ are defined similarly to the coefficients of the second fundamental form (2.6)

$$v = \langle\, \partial J\, K\,\rangle\,, \qquad\qquad \bar{v} = \left\langle\, \bar{\partial} J\, K\,\right\rangle\,. \tag{3.26}$$

The consistency conditions for the linear system (3.24),

$$\bar{\partial}\partial\beta + e^{-\beta}\, v\, \bar{v} = 0\,, \qquad \partial\bar{v} = 0\,, \qquad \bar{\partial}v = 0\,, \tag{3.27}$$

coincide with (2.7).

Due to the equivalence with the $\mathbb{R}^{1,2}$ case, we use the same gauge fixing conditions. The first corresponds to the parameterization

$$v(z) = f'(z)\,, \qquad \bar{v}(\bar{z}) = -\bar{f}'(\bar{z})\,, \qquad e^{\beta} = \frac{1}{2}\left[\bar{f}(\bar{z}) - f(z)\right]^2, \tag{3.28}$$

and the second to the Liouville gauge with

$$v(z) = \lambda\,, \qquad \bar{v}(\bar{z}) = \bar{\lambda}\,, \qquad \bar{\partial}\partial\beta + \lambda\lambda\, e^{-\beta}\,. \tag{3.29}$$

As we will see in the next subsection the conditions (3.28) describe the nilpotently gauged WZW theory.

On the level of solutions of the linear system, the pair $(J, \bar{J})$ is completely equivalent to $(B, \bar{B})$, and we can use the solutions (2.12) and (2.43).

The next step is a construction of string worldsheets $g(z, \bar{z})$. The WZW field splits into the product of chiral (left) and anti-chiral (right) fields

$$g(z, \bar{z}) = g_l(z)\, g_r(\bar{z})\,, \tag{3.30}$$

and to find the string worldsheet one has to integrate the equations

$$g_l'(z) = J(z)\, g_l(z)\,, \qquad g_r'(\bar{z}) = g_r(\bar{z})\, \bar{J}(\bar{z})\,. \tag{3.31}$$

Then the induced metric (3.21) can be calculated by

$$e^{\alpha} = -\left\langle\, g_l^{-1}(z)\, g_l'(z)\, g_r'(\bar{z}), g_r^{-1}(\bar{z}) \,\right\rangle . \tag{3.32}$$

We realize this programme in the following two subsections. Some helpful formulas for $\mathrm{SL}(2, \mathbb{R})$ calculations are presented in appendix G.

3.4 Nilpotent gauge

Let us consider the gauge (3.28). The isometry between $sl(2, \mathbb{R})$ and $\mathbb{R}^{1,2}$ relates the standard orthonormal bases of this spaces $t_\mu \leftrightarrow \mathbf{e}_\mu$, $\mu = (0, 1, 2)$. The basis (2.15) then corresponds to

$$\mathbf{e} \leftrightarrow t_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathbf{e}_+ \leftrightarrow t_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{e}_- \leftrightarrow t_- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \tag{3.33}$$

and similarly to (2.12) we obtain the Kac-Moody currents

$$J(z) = \begin{pmatrix} f(z) & 1 \\ -f^2(z) & -f(z) \end{pmatrix}, \qquad \bar{J}(\bar{z}) = \begin{pmatrix} \bar{f}(\bar{z}) & 1 \\ -\bar{f}^2(\bar{z}) & -\bar{f}(\bar{z}) \end{pmatrix}. \tag{3.34}$$

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The matrixes $t_\pm = \frac{1}{2}(t_0 \pm t_1)$ are nilpotent elements ($t_\pm^2 = 0$) of the $sl(2,\mathbb{R})$ algebra. The currents (3.34) have constant components in t_+ direction equal to 1 ($J^+ = 1 = \bar{J}^+$). Note that the transformation to the basis $(P\,\mathbf{e}_+,\, P^{-1}\,\mathbf{e}_-,\, \mathbf{e})$ used in (2.23) is equivalent to the rescaling of the t_+ and t_- components in (3.34) by P and P^{-1}, respectively.

It is well known that the nilpotent gauging of the SL$(2,\mathbb{R})$ WZW model leads to Liouville theory [28]. This gauging corresponds to the Hamiltonian reduction with constant $J^+(z)$ and $\bar{J}^+(\bar{z})$, similarly to (3.34). The Kac-Moody currents (3.34) satisfy the Virasoro constraints (3.22) as well. Constant $J^+(z)$, $\bar{J}^+(\bar{z})$ and the Virasoro constraints together form the second class constraints. Thus, these two sets of constraints are complementary to each other and they provide a coset construction.

Applying the reduction scheme used in coset WZW models, we write the chiral and antichiral parts of the WZW-field in a matrix form

$$g_l(z) = \begin{pmatrix} \psi(z) & \chi(z) \\ \xi(z) & \eta(z) \end{pmatrix}, \qquad g_r(\bar{z}) = \begin{pmatrix} \bar{\xi}(\bar{z}) & \bar{\psi}(\bar{z}) \\ \bar{\eta}(\bar{z}) & \bar{\chi}(\bar{z}) \end{pmatrix}, \tag{3.35}$$

and from (3.31) find the relations

$$\xi(z) = \psi'(z) - f(z)\,\psi(z)\,, \qquad \bar{\xi}(\bar{z}) = \bar{\psi}'(\bar{z}) + \bar{f}(\bar{z})\,\bar{\psi}(\bar{z})\,, \tag{3.36}$$

$$\eta(z) = \chi'(z) - f(z)\,\chi(z)\,, \qquad \bar{\eta}(\bar{z}) = \chi'(\bar{z}) + \bar{f}(\bar{z})\,\bar{\chi}(\bar{z})\,, \tag{3.37}$$

and the Hill equations

$$\psi''(z) = f'(z)\,\psi(z)\,, \qquad \bar{\psi}''(\bar{z}) = -\bar{f}'(\bar{z})\,\bar{\psi}(\bar{z})\,, \tag{3.38}$$

$$\chi''(z) = f'(z)\,\chi(z)\,, \qquad \bar{\chi}''(\bar{z}) = -\bar{f}'(\bar{z})\,\bar{\chi}(\bar{z})\,, \tag{3.39}$$

which are satisfied by the components of the first row of $g_l(z)$ and the second column of $g_r(\bar{z})$. These components are invariant under the gauge transformations generated by the nilpotent currents. The unimodularity conditions $\det g_l(z) = 1 = \det g_r(\bar{z})$ provide the unit Wronskians, as the normalization conditions for the solutions of (3.38)

$$\psi(z)\chi'(z) - \psi'(z)\,\chi(z) = 1\,, \qquad \bar{\psi}(\bar{z})\bar{\chi}'(\bar{z}) - \bar{\psi}'(\bar{z})\,\bar{\chi}(\bar{z}) = -1\,. \tag{3.40}$$

Thus, we get the WZW-field

$$g = \begin{pmatrix} \psi(z) & \chi(z) \\ \psi'(z) - f(z)\psi(z) & \chi'(z) - f(z)\chi(z) \end{pmatrix} \begin{pmatrix} \bar{\psi}'(\bar{z}) + \bar{f}(\bar{z})\bar{\psi}(\bar{z}) & \bar{\psi}(\bar{z}) \\ \bar{\chi}'(\bar{z}) + \bar{f}(\bar{z})\bar{\chi}(\bar{z}) & \bar{\chi}(\bar{z}) \end{pmatrix}, \tag{3.41}$$

parameterized by the gauge invariant chiral $(\psi,\, \chi)$ and antichiral $(\bar{\psi},\, \bar{\chi})$ fields, which are related by the unit Wronskians (3.40). The g_{12} matrix element of the WZW-field $g_{12}(z,\bar{z}) = \psi(z)\bar{\psi}(\bar{z}) + \chi(z)\bar{\chi}(\bar{z})$ is also gauge invariant and it is identified with the Liouville field exponent $V(z,\bar{z})$ of the conformal weight $-\frac{1}{2}$ (see appendix C). The potentials in the Hill equations form the stress tensor of Liouville theory

$$f'(z) = T(z)\,, \qquad -\bar{f}'(\bar{z}) = \bar{T}(\bar{z})\,. \tag{3.42}$$

On the other hand, the WZW-field (3.41) describes a string surface in $\mathrm{SL}(2,\mathbb{R})$. The worldsheet induced metric (3.32) obtained from (3.41) reads

$$e^{\alpha} = \left[\psi'(z)\,\bar{\psi}'\,(\bar{z}) + \chi'(z)\,\bar{\chi}'(\bar{z})\right]^2 . \tag{3.43}$$

In this way we parameterize the $\mathrm{SL}(2,\mathbb{R})$ string surfaces by the chiral and antichiral functions of Liouville theory. However, the fact that the components of the stress tensor (3.42) are given as derivatives of periodic functions imposes certain restrictions on allowed Liouville fields. Namely, the chiral and antichiral energy functionals of Liouville theory, given by the integral of the stress tensor over the period, have to vanish

$$L_0 = \int_0^{2\pi} \mathrm{d}z\; T(z) = 0\,, \qquad \bar{L}_0 = \int_0^{2\pi} \mathrm{d}\bar{z}\; \bar{T}(\bar{z}) = 0\,. \tag{3.44}$$

Note that solutions of the Hill equation with a periodic potential are only quasi-periodic. Writing the pairs $(\psi,\, \chi)$ and $\bar{\psi},\, \bar{\chi})$ as a row and column, respectively

$$\Psi^T = (\psi \quad \chi)\,, \qquad \bar{\Psi} = \begin{pmatrix} \bar{\psi} \\ \bar{\chi} \end{pmatrix}, \tag{3.45}$$

one gets $\Psi^T(z+2\pi) = \Psi^T(z)M$ and $\bar{\Psi}(\bar{z}+2\pi) = \bar{M}\bar{\Psi}(\bar{z})$, with $M \in \mathrm{SL}(2,\mathbb{R})$ and $\bar{M} \in \mathrm{SL}(2,\mathbb{R})$. The monodromies of the chiral WZW-fields in (3.41) then are given by

$$g_l(z+2\pi) = g_l(z)\,M\,, \qquad g_r(\bar{z}+2\pi) = \bar{M}\,g_r(\bar{z}) \tag{3.46}$$

and the periodicity of $g(\tau,\sigma)$ requires $M = \bar{M}$. The transformation $g_l(z) \mapsto g_l(z)N$, $g_r(\bar{z}) \mapsto N^{-1}g_r(\bar{z})$ leaves the general solution (3.30) invariant and transforms the monodromy matrix by $M \mapsto N^{-1}MN$. The monodromies with $|\langle M\rangle| < 1$, $|\langle M\rangle| = 1$ and $|\langle M\rangle| > 1$ are called elliptic, parabolic and hyperbolic, respectively. The monodromy properties of the chiral fields (ψ, χ) play an important role in the classification of Liouville fields in terms of the coadjoint orbits of the Virasoro algebra [16].

As an illustrative toy example of the scheme let's consider the case with constant $f(z)$ and $\bar{f}(\bar{z})$. They correspond to vanishing $T(z)$ and $\bar{T}(\bar{z})$, and the solutions of the Hill equations with the unit Wronskians are

$$\psi(z) = 1\,, \qquad \chi(z) = z\,; \qquad \bar{\psi}(\bar{z}) = \bar{z}\,, \qquad \bar{\chi}(\bar{z}) = 1\,. \tag{3.47}$$

These functions correspond to the parabolic monodromy with $M = I + 2\pi\, t_+$. Inserting them in (3.41) with $f(z) = c$ and $\bar{f}(\bar{z}) = \bar{c}$ one gets a WZW-field, which depends only on τ, and describes a particle trajectory like constant $f(z)$ and $\bar{f}(\bar{z})$ in $\mathbb{R}^{1,2}$.

In general, the Hill equation can not be integrated explicitly for an arbitrary potential. Therefore, in contrast to $\mathbb{R}^{1,2}$, the functions f and $\bar{f}$ are not convenient parameterizing variables for string surfaces in $\mathrm{SL}(2,\mathbb{R})$. Usually, it is more helpful to parameterize the functions $(\psi,\, \chi)$ and $(\bar{\psi}, \bar{\chi})$ directly and express f and $\bar{f}$ through them.

Let's consider the following class of chiral and antichiral fields

$$\psi(z) = \frac{\cos[\theta\zeta(z)]}{\sqrt{\theta\,\zeta'(z)}}\,, \qquad \bar\psi(\bar z) = \frac{\sin[\bar\theta\bar\zeta(\bar z)]}{\sqrt{\bar\theta\,\bar\zeta'(\bar z)}}\,,$$
$$\chi(z) = \frac{\sin[\theta\zeta(z)]}{\sqrt{\theta\,\zeta'(z)}}\,, \qquad \bar\chi(\bar z) = \frac{\cos[\bar\theta\bar\zeta(\bar z)]}{\sqrt{\bar\theta\,\bar\zeta'(\bar z)}}\,, \tag{3.48}$$

where $\zeta(z)$ and $\zeta(\bar z)$ are monotonic functions with the monodromies (2.10) and θ and $\bar\theta$ are positive numbers, which we specify below. The monodromies of these functions

$$M = \begin{pmatrix} \cos(2\pi\theta) & \sin(2\pi\theta) \\ -\sin(2\pi\theta) & \cos(2\pi\theta) \end{pmatrix}, \qquad \bar M = \begin{pmatrix} \cos(2\pi\bar\theta) & \sin(2\pi\bar\theta) \\ -\sin(2\pi\bar\theta) & \cos(2\pi\bar\theta) \end{pmatrix} \tag{3.49}$$

belong to the elliptic class and the periodicity requires $\bar\theta = \theta + m$, with an integer m.

Like in (2.49), eq. (3.48) can be considered as a Virasoro orbit of trigonometric functions, which are solutions of the Hill equations for $T(z) = -\theta^2$ and $\bar T(\bar z) = -\bar\theta^2$. The chiral stress tensor obtained from (3.48)

$$T(z) = \frac{\psi''(z)}{\psi(z)} = -\theta^2\zeta'^2(z) + \frac{1}{4}\left(\frac{\zeta''(z)}{\zeta'(z)}\right)^2 - \frac{1}{2}\left(\frac{\zeta''(z)}{\zeta'(z)}\right)'\,, \tag{3.50}$$

corresponds to the transformation law of $T(z)$ with the Schwarz derivative. Then, the parameter θ defined by

$$\theta^2\int_0^{2\pi} dz\,\zeta'^2(z) = \frac{1}{4}\int_0^{2\pi} dz\,\left(\frac{\zeta''(z)}{\zeta'(z)}\right)^2\,, \tag{3.51}$$

provides vanishing of the energy functional (3.44).

The induced metric (3.43) calculated from (3.48) reads

$$e^\alpha = \theta\bar\theta\,\zeta'(z)\bar\zeta'(\bar z)\Big(A(z,\bar z)\,\sin\left[\theta\zeta(z)+\bar\theta\bar\zeta(\bar z)\right] + B(z,\bar z)\,\cos\left[\theta\zeta(z)+\bar\theta\bar\zeta(\bar z)\right]\Big)^2\,, \tag{3.52}$$

where

$$A(z,\bar z) = 1 - \frac{\zeta''(z)\,\bar\zeta''(\bar z)}{4\theta\bar\theta\,\zeta'^{\,2}(z)\,\bar\zeta'^{\,2}(\bar z)}\,, \qquad B(z,\bar z) = \frac{\zeta''(z)}{2\theta\,\zeta'^{\,2}(z)} + \frac{\bar\zeta''(\bar z)}{2\bar\theta\,\bar\zeta'^{\,2}(\bar z)}\,. \tag{3.53}$$

With the parameterization $A = 1 - \tan\gamma(z)\,\tan\bar\gamma(\bar z)$ and $B = \tan\gamma(z) + \tan\bar\gamma(\bar z)$, the conformal factor (3.52) takes the form

$$e^\alpha = \theta\,\bar\theta\,\zeta'(z)\,\bar\zeta'(\bar z)\,R^2\,\sin^2\left[\theta\,\zeta(z) + \bar\theta\,\bar\zeta(\bar z) + \gamma(z) + \bar\gamma(\bar z)\right]\,, \tag{3.54}$$

where $R^2 = A^2 + B^2 = \left[1+\tan^2\gamma(z)\right]\left[1+\tan^2\bar\gamma(\bar z)\right]$. The solutions of the equation

$$\theta\,\zeta(z) + \bar\theta\,\bar\zeta(z) + \gamma(z) + \bar\gamma(\bar z) = n\pi\,, \tag{3.55}$$

define the worldsheet singular points. Since the chiral and antichiral parts of $B(z,\bar z)$ are periodic functions, the introduced angle variables are bounded by $-\frac{\pi}{2} < \gamma(z) < \frac{\pi}{2}$. The functions $\zeta(z)$ and $\bar\zeta(z)$, in contrast, are unbounded. Hence, eq. (3.55) always has solutions and the string surfaces are singular.

3.5 Liouville gauge for $\mathrm{SL}(2,\mathbb{R})$ string

Now we consider the Liouville gauge (3.29) and the surfaces related to the ground state functions (2.47). As in the previous subsection, we use the correspondence between the worldsheet tangent vectors in $\mathbb{R}^{1,2}$ and the $\mathrm{SL}(2,\mathbb{R})$ Kac-Moody currents.

The tangent vectors (2.50) are equivalent to the currents

$$J(z) = \Lambda\,[t_0 + \cos(nz)\,t_1 + \sin(nz)\,t_2]\,, \qquad \bar J(\bar z) = \bar\Lambda\,[t_0 - \cos(\bar n\bar z)\,t_1 + \sin(\bar n\bar z)\,t_2]\,. \tag{3.56}$$

Note that the periodicity condition here does not require necessarily $\Lambda = \bar\Lambda$. However, if $\Lambda = \bar\Lambda$, the currents (3.56) have equal constant t_0 components $J_0 = \Lambda = \bar J_0$. These constraints correspond to the vector gauged $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset model [29]. A more well investigated case is the axial gauged $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ model [30], which corresponds to $J_0 = \Lambda = -\bar J_0$. This model was integrated in [31] similarly to Liouville theory, and in the periodic case a free-field parameterization was obtained in the hyperbolic sector. The exact integrability of the model has been generalized in [32] to the vector gauged model and to the elliptic sector of both models as well. We use the corresponding technique to integrate the Kac-Moody currents (3.56) to WZW-fields.

The currents (3.56) can be written in the form (see (G.4))

$$J(z) = \Lambda\, e^{\frac{1}{2}\,nz\,t_0}\,(t_0 + t_1)\,e^{-\frac{1}{2}\,nz\,t_0}\,, \qquad \bar J(\bar z) = \bar\Lambda\, e^{-\frac{1}{2}\,\bar n\bar z\,t_0}\,(t_0 - t_1)\,e^{\frac{1}{2}\,\bar n\bar z\,t_0}\,, \tag{3.57}$$

and with $h_l(z) = e^{-\frac{1}{2}\,nz\,t_0}\,g_l(z)$ and $h_r(\bar z) = g_r(\bar z)\,e^{-\frac{1}{2}\,\bar n\bar z\,t_0}$, eq. (3.31) reduces to

$$h_l'(z) = \left[\left(\Lambda - \frac{n}{2}\right)t_0 + \Lambda t_1\right]h_l(z)\,, \qquad h_r'(\bar z) = h_r(\bar z)\left[\left(\bar\Lambda - \frac{\bar n}{2}\right)t_0 - \bar\Lambda t_1\right]. \tag{3.58}$$

The integration is then straightforward and leads to

$$g(z,\bar z) = e^{\frac{1}{2}\,nz\,t_0}\,e^{\frac{1}{2}\,z\,a}\,g_0\,e^{\frac{1}{2}\,\bar z\,\bar a}\,e^{\frac{1}{2}\,\bar n\bar z\,t_0}\,, \tag{3.59}$$

where g_0 is a $\mathrm{SL}(2,\mathbb{R})$ valued integration constant and

$$a = (2\Lambda - n)\,t_0 + 2\Lambda\,t_1\,, \qquad \bar a = \left(2\bar\Lambda - \bar n\right)t_0 - 2\bar\Lambda\,t_1\,. \tag{3.60}$$

The periodicity of (3.59) imposes the following condition

$$g_0^{-1}\,e^{\pi a}\,g_0 = (-)^{n-\bar n}\,e^{\pi\bar a}\,. \tag{3.61}$$

Equations (3.59)–(3.61) define the WZW-fields $g(z,\bar z)$ and, thereby, the string surfaces in $\mathrm{SL}(2,\mathbb{R})$. However, to describe the structure of these surfaces in detail, similarly to the flat case, an additional labour is needed.

Let us consider timelike a and $\bar a$. In this case (see (G.1)) the exponents in (3.61) take the form

$$e^{\pi\,a} = \cos(\pi\theta)\;I + \sin(\pi\theta)\;\hat a\,, \qquad e^{\pi\,\bar a} = \cos\left(\pi\bar\theta\right)\;I + \sin\left(\pi\bar\theta\right)\;\hat{\bar a}\,, \tag{3.62}$$

where

$$\theta = \sqrt{|\langle\, a\, a\,\rangle|}\,, \qquad \bar\theta = \sqrt{|\langle\,\bar a\,\bar a\,\rangle|}\,, \tag{3.63}$$

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and $\hat{a}$, $\hat{\bar{a}}$ are the normalized $sl(2,\mathbb{R})$ matrixes

$$\hat{a} = \frac{a}{\theta} = \begin{pmatrix} 0 & -\frac{\theta}{n} \\ \frac{n}{\theta} & 0 \end{pmatrix}, \qquad \hat{\bar{a}} = \frac{\bar{a}}{\bar{\theta}} = \begin{pmatrix} 0 & -\frac{\bar{n}}{\bar{\theta}} \\ \frac{\bar{\theta}}{\bar{n}} & 0 \end{pmatrix}. \tag{3.64}$$

From (3.61) and (3.62) follows the equation for g_0

$$g_0^{-1}\,\hat{a}\,g_0 = s\,\hat{\bar{a}}\,, \tag{3.65}$$

and the relations between θ and $\bar{\theta}$

$$\cos(\pi\theta) = \epsilon\cos\left(\pi\bar{\theta}\right), \qquad \sin(\pi\theta) = s\,\epsilon\sin\left(\pi\bar{\theta}\right). \tag{3.66}$$

Here $\epsilon = (-)^{n-\bar{n}}$ and $s = \pm 1$, since the similarity transformation $\hat{a} \mapsto g_0^{-1}\,\hat{a}\,g_0$ preserves the norm of $\hat{a}$. Another invariant of the transformation $\hat{a} \mapsto g_0^{-1}\,\hat{a}\,g_0$ is the sign of $\hat{a}^0$, where $\hat{a}^0$ is the t_0 component of $\hat{a}$. The t_0 components of the unit vectors (3.64) are

$$\hat{a}^0 = -\frac{\theta^2+n^2}{2n\theta}\,, \qquad \hat{\bar{a}}^0 = -\frac{\bar{\theta}^2+\bar{n}^2}{2\bar{n}\bar{\theta}}\,, \tag{3.67}$$

and we conclude that $s = \mathrm{sign}(n\,\bar{n})$.

The solution of (3.65) is given by

$$g_0 = \frac{1}{\sqrt{s\,n\,\bar{n}\,\theta\bar{\theta}}}\begin{pmatrix} \theta\,\bar{\theta}\,\cos\phi & -\,\theta\,\bar{n}\,\sin\phi \\ s\,n\,\bar{\theta}\,\sin\phi & s\,n\,\bar{n}\,\cos\phi \end{pmatrix}, \tag{3.68}$$

where ϕ is an angle variable, which parameterize the freedom in g_0. In addition, eqs. (3.65) and (3.59) yield

$$g(z,\bar{z}) = e^{\frac{1}{2}\,n z\,t_0}\,e^{\frac{1}{2}\,(\theta z + s\,\bar{\theta}\bar{z})\,\hat{a}}\,g_0\,e^{\frac{1}{2}\,\bar{n}\bar{z}\,t_0}\,. \tag{3.69}$$

To visualize the constructed worldsheets (3.69), we introduce two complex planes defined by the embedding coordinates (3.9)

$$Z = Y_1 + iY_2 = \langle (t_1 + it_2)\,g\,\rangle\,, \qquad Z^0 = Y^{\tilde{0}} + iY^0 = \langle (I - it_0)\,g\,\rangle\,. \tag{3.70}$$

Taking into account that t_0 is the generator of rotations in (t_1, t_2) and (I, t_0) planes (see (G.5)–(G.6)), from (3.69) we obtain

$$\begin{aligned} Z &= e^{\frac{i}{2}\,(nz-\bar{n}\bar{z})}\left[A_+\,e^{\frac{i}{2}\,(\theta z+s\,\bar{\theta}\bar{z})} + A_-\,e^{-\frac{i}{2}\,(\theta z+s\,\bar{\theta}\bar{z})}\right], \\ Z^0 &= e^{\frac{i}{2}\,(nz+\bar{n}\bar{z})}\left[B_+\,e^{\frac{i}{2}\,(\theta z+s\,\bar{\theta}\bar{z})} + B_-\,e^{-\frac{i}{2}\,(\theta z+s\,\bar{\theta}\bar{z})}\right]. \end{aligned} \tag{3.71}$$

The coefficients $A_\pm$ and $B_\pm$ correspond to the following normalized traces

$$\begin{aligned} A_\pm &= \frac{1}{2}\langle\,(t_1+it_2)\,(I \mp i\hat{a})\,g_0\,\rangle = i\,\frac{s\,(\theta \mp n)\left(s\,\bar{\theta} \pm \bar{n}\right)\,e^{\pm is\phi}}{4\sqrt{s\,n\bar{n}\,\theta\bar{\theta}}}\,, \\ B_\pm &= \frac{1}{2}\langle\,(I - it_0)\,(I \mp i\hat{a})\,g_0\,\rangle = \frac{s\,(\theta \mp n)\left(s\,\bar{\theta} \mp \bar{n}\right)\,e^{\pm is\phi}}{4\sqrt{s\,n\bar{n}\,\theta\bar{\theta}}}\,. \end{aligned} \tag{3.72}$$

The conformal factor of the induced metric tensor obtained from these equations reads

$$e^{\alpha} = \mathrm{Re}\left(\partial Z_0\,\bar\partial Z_0^* - \partial Z\,\bar\partial Z^*\right) = \frac{s\,\left(\theta^2 - n^2\right)\left(\bar\theta^2 - \bar n^2\right)}{16\theta\bar\theta}\,\left[1 + \cos\left(\theta z + s\,\bar\theta\bar z + 2s\,\phi\right)\right]\,. \tag{3.73}$$

According to (3.63) and (3.60), the parameters of the solutions are related by

$$\theta^2 = n^2 - 4\Lambda\,n\,, \qquad \bar\theta^2 = \bar n^2 - 4\bar\Lambda\,\bar n\,. \tag{3.74}$$

These relations reduce the conformal factor (3.73) to

$$e^{\alpha} = \frac{\Lambda\,\bar\Lambda\,|n\,\bar n|}{\theta\bar\theta}\,\left[1 + \cos\left(\theta z + s\,\bar\theta\bar z + 2s\,\phi\right)\right]\,. \tag{3.75}$$

Since Λ, $\bar\Lambda$, θ, $\bar\theta$ are positive, the metric is regular almost everywhere. The singular points correspond to the solutions of the equation $1 + \cos(\theta z + s\,\bar\theta\bar z + 2s\,\phi) = 0$, i.e.

$$\left(\theta + s\,\bar\theta\right)\tau + \left(\theta - s\,\bar\theta\right)\sigma = (2m+1)\pi - 2s\,\phi \qquad (m \in \mathbb{Z})\,. \tag{3.76}$$

Note that eqs. (3.75)–(3.76) are similar to (2.57)–(2.58), derived for the spiky and circular strings in $\mathbb{R}^{1,2}$.

Now we use the relation (3.66) to exclude one continuous parameter. The case $\epsilon = 1$ corresponds to even $n \pm \bar n$. According to (3.66), $\theta - s\,\bar\theta$ is also even for $\epsilon = 1$. With the notations

$$\theta - s\,\bar\theta = 2\nu\,, \qquad \theta + s\,\bar\theta = 2\mu\,, \qquad n - \bar n = 2k\,, \qquad n + \bar n = 2l\,, \tag{3.77}$$

the solution (3.71) can be written as

$$\begin{aligned} Z(\tau,\sigma) &= e^{i(k\tau + l\sigma)}\left[A_+\,e^{i(\mu\tau+\nu\sigma)} + A_-\,e^{-i(\mu\tau+\nu\sigma)}\right]\,, \\ Z^0(\tau,\sigma) &= e^{i(l\tau + k\sigma)}\left[B_+\,e^{i(\mu\tau+\nu\sigma)} + B_-\,e^{-i(\mu\tau+\nu\sigma)}\right]\,. \end{aligned} \tag{3.78}$$

The parameters $(\nu,\,k,\,l)$ here are integer, which provides the periodicity of (3.78).

If $\epsilon = -1$, $n \pm \bar n$ and $\theta - s\,\bar\theta$ are odd. Therefore, instead of (3.77) we use the notations

$$\theta - s\,\bar\theta = 2\nu + 1\,, \qquad \theta + s\,\bar\theta = 2\mu + 1\,, \qquad n - \bar n = 2k + 1\,, \qquad n + \bar n = 2l + 1\,, \tag{3.79}$$

again with integer ν, k and l. The solution (3.71) in this case becomes

$$\begin{aligned} Z(\tau,\sigma) &= e^{i(k\tau + l\sigma)}\left[A_+\,e^{i[(\mu+1)\tau+(\nu+1)\sigma]} + A_-\,e^{-i(\mu\tau+\nu\sigma)}\right]\,, \\ Z^0(\tau,\sigma) &= e^{i(l\tau + k\sigma)}\left[B_+\,e^{i[(\mu+1)\tau+(\nu+1)\sigma]} + B_-\,e^{-i(\mu\tau+\nu\sigma)}\right]\,. \end{aligned} \tag{3.80}$$

One has to remember that the parameters of the solutions are restricted by

$$n \neq 0\,, \quad \bar n \neq 0\,; \quad \theta > 0\,, \quad \bar\theta > 0; \quad \theta \neq |n|\,, \quad \bar\theta \neq |\bar n|\,, \tag{3.81}$$

where the last two inequalities follow from (3.74). If these conditions are not fulfilled, the coefficients $A_\pm$ and $B_\pm$ are either singular, or vanishing. The vanishing of the coefficients correspond to the degenerated case $\Lambda = 0 = \bar{\Lambda}$. The formulation of the conditions (3.81) in terms of the new parameters (μ, ν, k, l) is more complicated. Therefore, sometimes it is more convenient to keep the old parameters (n, $\bar{n}$, θ, $\bar{\theta}$).

The spatial part of AdS_3 is given by the complex plane Z. Due to the similarity with the flat space, the function Z in (3.71) can be described in a same manner as (2.56) for $\mathbb{R}^{1,2}$ strings. The case $\theta - s\,\bar{\theta} = 0$ is special, like $n - \bar{n} = 0$ in $\mathbb{R}^{1,2}$, and we consider it here separately.

The condition $\theta - s\,\bar{\theta} = 0$ implies $s = 1$ and $\theta = \bar{\theta}$. The corresponding solutions of eq. (3.76) are σ-independent discrete values of τ, like for the circular oscillating strings in $\mathbb{R}^{1,2}$. For simplicity, let's assume $n = \bar{n}$ and $2\phi = -\pi$. The solution (3.71)–(3.72) then reduces to

$$Z = \frac{n^2-\theta^2}{2i\,n\,\theta}\,\sin(\theta\tau)\,e^{in\sigma}\,, \qquad Z^0 = \left(\frac{n^2+\theta^2}{2n\,\theta}\,\sin(\theta\tau) + i\cos(\theta\tau)\right)\,e^{in\tau}\,, \tag{3.82}$$

and the conformal factor (3.73) becomes

$$e^{\alpha} = \frac{\left(n^2-\theta^2\right)^2}{8\theta^2}\,\sin^2(\theta\tau)\,. \tag{3.83}$$

The time variable in AdS_3 is given by the phase of Z^0, which in (3.82) is only τ dependent. For a fixed τ, the function Z in (3.82) provides a circle with the radius proportional to $\sin(\theta\tau)$. Therefore, eq. (3.83) describes a circular oscillating string like (2.59) in $\mathbb{R}^{1,2}$. This type of string solutions were obtained earlier in [33], where the authors used the Pohlmeyer type scheme for the embedding space $\mathbb{R}^{2,2}$, and provided some non periodic solutions as well.

In the limit $\theta \to 0$ eqs. (3.82)–(3.83) are reduced to

$$Z = \frac{n\,\tau}{2i}\,e^{in\sigma}\,, \qquad Z^0 = \frac{1}{2}\,(n\tau + 2i)\,e^{in\tau}\,; \qquad e^{\alpha} = \frac{n^4}{8}\,\tau^2\,. \tag{3.84}$$

This case correspond to a nilpotent a in (3.64), and the related WZW-field belongs to the parabolic monodromy.

Eqs. (3.82)–(3.83) allow a continuation to imaginary θ, which leads to the hyperbolic solutions with

$$Z = \frac{n^2+\theta^2}{2i\,n\theta}\,\sinh(\theta\tau)\,e^{in\sigma}\,, \qquad Z^0 = \left(\frac{n^2+\theta^2}{2n\,\theta}\,\sinh(\theta\tau) + i\cosh(\theta\tau)\right)\,e^{in\tau}\,, \tag{3.85}$$

and the conformal factor

$$e^{\alpha} = \frac{(\theta^2+n^2)^2}{8\theta^2}\,\sinh^2(\theta\tau)\,. \tag{3.86}$$

These parabolic and hyperbolic solutions shrink to the origin $Z = 0$ only once at $\tau = 0$. One can show that other parabolic and hyperbolic solutions corresponding to $n \neq \bar{n}$ are also obtained by the analytical continuation of the elliptic solutions (3.71) with $s = 1$ and $\theta = \bar{\theta}$.

Let's assume now that $\theta - s\,\bar{\theta} \neq 0$. The function $Z(\tau,\sigma)$ in (3.71) then fulfills the relation (2.60) with

$$\omega = -\frac{\theta\,\bar{n} + n\,s\,\bar{\theta}}{\theta - s\,\bar{\theta}}\,, \qquad \omega_0 = \frac{\theta + s\,\bar{\theta}}{\theta - s\,\bar{\theta}}\,, \tag{3.87}$$

and the initial configuration

$$Z(\sigma) = \left[A_+ \, e^{\frac{i}{2}\,(n+\bar{n}+\theta-s\,\bar{\theta})\sigma} + A_- \, e^{\frac{i}{2}\,(n+\bar{n}-\theta z+s\,\bar{\theta}\bar{)}\sigma}\right] . \tag{3.88}$$

As in the flat case, the dynamics in τ preserves the shape of this closed curve. Effectively it rotates only. The curve defined by (3.88) is represented again as a combination of two rotations. Therefore, the properties of the spiky strings, discussed in Subsection 2.3 and appendix E, can be generalized to this case.

Finally, we briefly describe the general case with the Kac-Moody current

$$J(z) = \frac{\Lambda}{\zeta'(z)}\, \left[t_0 + \cos(n\zeta(z))\, t_1 + \sin(n\zeta(z))\, t_2\right] , \tag{3.89}$$

which corresponds to the tangent vector (2.69). The description of the antichiral part is similar. Using the same trick as in (3.57), eq. (3.31) reduces to

$$h_l'(z) = \left[\frac{\Lambda}{\zeta'(z)}\,(t_0+t_1) - \frac{n\,\zeta'(z)}{2}\, t_0\right]\, h_l(z)\,, \tag{3.90}$$

with $h_l(z) = e^{-\frac{1}{2}\,n\zeta(z)\,t_0}\, g_l(z)$. The change of variable $z \mapsto \zeta$ in (3.90) yields

$$\partial_\zeta h_l(\zeta) = \left[\Lambda\rho^2(\zeta)\,(t_0+t_1) - \frac{n}{2}\, t_0\right]\, h_l(\zeta)\,, \tag{3.91}$$

where $\rho(\zeta) = \left(\frac{dz}{d\zeta}\right)^2$. The chiral current, which stands in this equation has vanishing t_2 and constant t_- components. Therefore, it can be interpreted as a current of nilpotently gauged WZW theory in the gauge $J_2 = 0$. This reduction is described again by Liouville theory and one can use the scheme of the previous subsection.

4 Summary

Here we give a summary of the paper and describe some unsolved problems. In summary we list the following items:

1. On the basis of the isometry between $\mathbb{R}^{1,2}$ and $sl(2,\mathbb{R})$, the Pohlmeyer scheme for string dynamics in $\mathbb{R}^{1,2}$ and $\mathrm{SL}(2,\mathbb{R})$ is formulated in equivalent forms. This equivalence maps the tangent vectors of the $\mathbb{R}^{1,2}$ string surfaces to the $\mathrm{SL}(2,\mathbb{R})$ Kac-Moody currents, which obey the Virasoro constraints.

2. The closed string dynamics in $\mathbb{R}^{1,2}$ is integrated within the Pohlmeyer scheme, using the parameterization (2.11) for the components of the fundamental quadratic forms. These parameterization fixes the conformal gauge freedom up to translations and a one parameter subgroup. The factorization of the set of solutions by the remaining conformal freedom provides the string surfaces in the lightcone gauge. These surfaces have a degenerated induced metric and the chiral components of the second fundamental form $u(z)$, $\bar{u}(\bar{z})$ do not have fixed signs in the interval of periodicity.

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3. The second class of closed string surfaces in $\mathbb{R}^{1,2}$ corresponds to the case with non vanishing $u(z)$ and $\bar{u}(\bar{z})$. In the gauge where $u(z)$ and $\bar{u}(\bar{z})$ are constant, the Gauß equation reduces to the Liouville equation and, on the basis of its general solution, the linear system of the Pohlmeyer scheme is integrated in the form (2.40) and (2.43). The periodicity condition is satisfied by the class of singular Liouville fields with the monodromy matrix $\pm I$. This class is parameterized by the Virasoro coadjoint orbits, which are labeled by two nonzero integers $(n, \bar{n})$. The signs of n and $\bar{n}$ coincide with the signs of u and $\bar{u}$, respectively. The vacuum Liouville fields with constant stress tensor $T(z) = -\frac{n^2}{4}$, $\bar{T}(\bar{z}) = -\frac{\bar{n}^2}{4}$ describe oscillating circular (if $n = \bar{n}$) and rotating spiky (if $n \neq \bar{n}$) strings. The number of spikes is equal to $|n - \bar{n}|$ and the shape of string configurations at a fixed time essentially depends on the sign of $n\bar{n}$.

4. The tangent vectors of $\mathbb{R}^{1,2}$ string surfaces in the lightcone gauge correspond to Kac-Moody currents of a nilpotently gauged $\mathrm{SL}(2,\mathbb{R})$ WZW model. The related Liouville fields are singular and have vanishing chiral energy functional. The corresponding string surfaces are described by a pair of monotonic functions $\zeta(z)$, $\bar{\zeta}(\bar{z})$, used in parameterization of 2d conformal group. These $\mathrm{SL}(2,\mathbb{R})$ string surfaces are always singular, like the surfaces in $\mathbb{R}^{1,2}$.

5. In the Liouville gauge, the vacuum field configurations provide the coset Kac-Moody currents of the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ model. They are labeled by two nonzero integers n and $\bar{n}$. The integration of the currents in the elliptic sector of WZW-fields leads to circular and spiky strings. These string surfaces split in four classes, depending on the sign of $n\bar{n}$ and the parity of $n - \bar{n}$. The parabolic and hyperbolic solutions are obtained by the analytical continuation of the elliptic solutions with $n\bar{n} > 0$ and even $n - \bar{n}$.

The main result of the paper is the description of the new classes of closed string solutions in $\mathbb{R}^{1,2}$ and $\mathrm{SL}(2,\mathbb{R})$. They are given by (2.69) in $\mathbb{R}^{1,2}$, and by (3.41), (3.48) and (3.71)–(3.72) in $\mathrm{SL}(2,\mathbb{R})$.

The construction of quantum theory of these solutions requires quantization of singular Liouville fields. The quantized Liouville theory on a cylindrical spacetime [34] and on its Euclidean counterpart [35] are one of the most remarkable results in 2d QFT. However, this theory corresponds to the quantization of regular Liouville fields, which allow a free-field parameterization. This parameterization is a basis for canonical quantization in the Minkowskian spacetime, where the parameterizing field can be chosen as the *in* (or *out*) field of the theory. It also helps to construct the vertex operators [36] and calculate their causal algebra [37].

The singular Liouville fields, we are interested in, have a regular stress tensor. A natural way for a generalization of the canonical scheme to the singular case is to find a free-field parameterization of these singular fields. Note that such Liouville fields on a plane allow a free-field parameterization. Namely, a Liouville field with N lines of singularities can be canonically parameterized by one regular free field and the asymptotic data of N relativistic particles [38]. Unfortunately, a generalization of this result to the periodic case is still unknown (see however [22]).

The vacuum Liouville field configurations, used in the spiky strings, for $n = 1$ correspond to Möbius invariant ground state, which arises in boundary Liouville theory on a strip with Dirichlet conditions. The Hamiltonian description of such field configurations was considered in [39], and was an attempt to understand Liouville theory with Dirichlet boundary conditions as a limit of the theory with Neumann conditions [40]. The latter correspond to the elliptic monodromy [41] and its Euclidean version is given by FZZT branes [42]. This programme also needs further investigation. A possible candidate for quantum theory of the spiky strings in $\mathbb{R}^{1,2}$ could be the 'Wick rotated' ZZ branes [43].

Another possible approach to quantum theory of singular Liouville fields is the geometric quantization on the Virasoro coadjoint orbits [44]. These orbits parameterize nilpotently gauged strings in $\mathrm{SL}(2,\mathbb{R})$ and spiky strings both in $\mathbb{R}^{1,2}$ and $\mathrm{SL}(2,\mathbb{R})$.

There is a renewed interest to Liouville theory caused by [45], and it is a challenge to understand whether the singular Liouville fields and spiky strings described in the present paper have any relation to it.

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A Integration of the Pohlmeyer scheme for $\mathbb{R}^{1,2}$ strings

In this appendix we integrate the linear system (2.5) with u, $\bar{u}$ and e^{α} given by (2.11). We then, briefly consider the case of the Liouville gauge, discussed in subsection 2.3.

Starting with the last equations in (2.5)

$$\partial N = \frac{2f'(z)}{\left[\bar{f}(\bar{z}) - f(z)\right]^2}\,\bar{B}(\bar{z})\,, \qquad \bar{\partial} N = -\frac{2\bar{f}'(\bar{z})}{\left[\bar{f}(\bar{z}) - f(z)\right]^2}\,B(z)\,, \tag{A.1}$$

one obtains

$$N = \frac{2\,\bar{B}(\bar{z})}{\bar{f}(\bar{z}) - f(z)} + \bar{b}(\bar{z}) = \frac{2\,B(z)}{\bar{f}(\bar{z}) - f(z)} + b(z)\,, \tag{A.2}$$

where $b(z)$ and $\bar{b}(\bar{z})$ are chiral and antichiral fields with values in $\mathbb{R}^{1,2}$. Multiplying (A.2) by the denominator $\bar{f}(\bar{z}) - f(z)$ and then differentiating it by $\bar{\partial}$ and ∂, one gets

$$b(z) = c\,f(z) + c_1\,, \qquad \bar{b}(\bar{z}) = -c\,\bar{f}(\bar{z}) + \bar{c}_1\,, \tag{A.3}$$

with $\mathbb{R}^{1,2}$-valued constant vectors c, c_1, $\bar{c}_1$. The insertion of (A.2)–(A.3) into the first equation of (2.5) leads to

$$B'(z) = c_1\,f'(z) + c\,f(z)\,f'(z)\,, \tag{A.4}$$

which is integrated to

$$B(z) = c_1\, f(z) + \frac{c}{2}\, f^2(z) + c_2\,, \tag{A.5}$$

where c_2 is a new $\mathbb{R}^{1,2}$-valued integration constant. The antichiral sector similarly yields

$$\bar B(\bar z) = -\bar c_1\, \bar f(\bar z) + \frac{c}{2}\, \bar f^2(\bar z) + \bar c_2\,. \tag{A.6}$$

With these $B(z)$ and $\bar B(\bar z)$ eq. (A.2) relates the integration constants in the chiral and antichiral sectors by $\bar c_1 = -c_1$, $\bar c_2 = c_2$. Then, (A.2) reads

$$N(z,\bar z) = \frac{\bar f(\bar z) + f(z)}{\bar f(\bar z) - f(z)}\, c_1 + \frac{f(z)\, \bar f(\bar z)}{\bar f(\bar z) - f(z)}\, c + \frac{2}{\bar f(\bar z) - f(z)}\, c_2\,. \tag{A.7}$$

Finally, using the notations $c_1 \equiv \mathbf{e}$, $c \equiv 2\mathbf{e}_-$ and $c_2 \equiv \mathbf{e}_+$, one obtains B, $\bar B$ and N in the form (2.12)–(2.13).

In the Liouville gauge, where u and $\bar u$ are constants and e^α is given by (2.39), we start again with the last equations in (2.5)

$$\partial N = \frac{2\epsilon}{|\bar\lambda|}\, \frac{F'(z)\, \bar F'(\bar z)}{\left[\epsilon\, F(z) + \bar\epsilon\, \bar F(\bar z)\right]^2}\, \bar B(\bar z)\,, \qquad \bar\partial N = \frac{2\bar\epsilon}{|\lambda|}\, \frac{F'(z)\, \bar F'(\bar z)}{\left[\epsilon\, F(z) + \bar\epsilon\, \bar F(\bar z)\right]^2}\, B(z)\,. \tag{A.8}$$

The integration steps here are as before. At the final stage one gets the following relations for the integration constants $\bar c_1 = -c_1$, $\bar\epsilon\, \bar c_2 = \epsilon\, c_2$, and with the notations $c_1 \equiv \mathbf{e}$, $c \equiv 2\mathbf{e}_-$ and $c_2 \equiv \epsilon \mathbf{e}_+$ one arrives at (2.40)–(2.41).

B Conformal freedom and the lightcone gauge

Here we analyze the freedom of conformal transformations in the gauge (2.11). Let's write these transformations in the infinitesimal form

$$\zeta(z) = z + \varepsilon\phi(z)\,, \qquad \bar\zeta(\bar z) = \bar z + \varepsilon\bar\phi(\bar z)\,, \tag{B.1}$$

where $\phi(z)$ and $\bar\phi(\bar z)$ are periodic functions and ε is an infinitesimal parameter. Keeping the first order terms in ε, from (2.8)–(2.9) and (2.11) we find

$$(\bar f - f)^2 \mapsto (\bar f - f)^2 + \varepsilon \left[\left(\bar f - f\right)^2 \left(\phi' + \bar\phi'\right) + 2\left(\bar f - f\right)\left(\bar f'\,\bar\phi - f'\,\phi\right)\right], \tag{B.2}$$

$$u \mapsto u + \varepsilon\left(2\phi'\, f' + \phi\, f''\right)\,, \qquad \bar u \mapsto \bar u - \varepsilon\left(2\bar\phi'\,\bar f' + \bar\phi\,\bar f''\right)\,. \tag{B.3}$$

These transformations preserve the gauge (2.11), if they correspond to infinitesimal shifts $f \mapsto f + \varepsilon\rho$ and $\bar f \mapsto \bar f + \varepsilon\bar\rho$, with some chiral $\rho(z)$ and antichiral $\bar\rho(\bar z)$ functions. By (B.2) one then finds the relation

$$2(\bar\rho - \rho) = \left(\bar f - f\right)\left(\phi' + \bar\phi'\right) + 2\left(\bar f'\,\bar\phi - f'\,\phi\right)\,. \tag{B.4}$$

Its differentiation in z and $\bar z$ leads to the equation $\bar f'\,\phi'' - f'\,\bar\phi'' = 0$, which is solved by

$$\phi''(z) = c\, f'(z)\,, \qquad \bar\phi''(\bar z) = c\, \bar f'(\bar z)\,, \tag{B.5}$$

with a constant c. The integration of this equation in two steps provides

$$\phi'(z) = c\,(f(z) - p)\,, \qquad \bar{\phi}'(\bar{z}) = c\,\left(\bar{f}(\bar{z}) - p\right)\,, \tag{B.6}$$

and

$$\phi(z) = \phi_0 + i\,c\sum_{n\neq 0}\frac{a_n}{n}\,e^{-inz}\,, \qquad \bar{\phi}(\bar{z}) = \bar{\phi}_0 + i\,c\sum_{n\neq 0}\frac{\bar{a}_n}{n}\,e^{-in\bar{z}}\,. \tag{B.7}$$

Here we have used the Fourier mode expansions (2.18) for $f(z)$ and $\bar{f}(\bar{z})$, the equality of their zero modes (2.19) and the periodicity of ϕ and $\bar{\phi}$.

From (B.6) and (B.4) we also find

$$\rho = \frac{1}{2c}\,\phi'^{\,2} + \frac{1}{c}\,\phi''\,\phi\,, \qquad \bar{\rho} = \frac{1}{2c}\,\bar{\phi}'^{\,2} + \frac{1}{c}\,\bar{\phi}''\,\bar{\phi}\,. \tag{B.8}$$

Note that $c = 0$ corresponds to translations $\phi(z) = \phi_0$, $\bar{\phi}(\bar{z}) = \bar{\phi}_0$, with constant ϕ_0 and $\bar{\phi}_0$. In this case (B.4) provides $\rho(z) = \phi_0\, f'(z)$ and $\bar{\rho}(\bar{z}) = \bar{\phi}_0\, \bar{f}'(\bar{z})$.

Eqs. (B.8) and (B.5) allow to write (B.3) in the form $u \mapsto u + \varepsilon\,\rho'$, $\bar{u} \mapsto \bar{u} + \varepsilon\,\bar{\rho}'$. This form of transformations corresponds to $f \mapsto f + \varepsilon\rho$, $\bar{f} \mapsto \bar{f} + \varepsilon\bar{\rho}$, and proves that the described conformal transformations preserve the gauge (2.11).

Finally we show that the conditions (2.19) are invariant under these conformal transformations. First recall that the zero mode of a periodic function is given by the integral of this function over the period. Since f and $\bar{f}$, as well as f^2 and $\bar{f}^2$, have equal zero modes, from (B.6) follows that the zero modes of $\phi'^{\,2}$ and $\bar{\phi}'^{\,2}$ are also equal. Then, due to (B.8), ρ and $\bar{\rho}$ have equal zero modes too. This proves the invariance of the first relation in (2.19).

To prove the second relation, one has to compare the zero modes of $f\,\rho$ and $\bar{f}\,\bar{\rho}$ (the first order terms in ε). Here, it is convenient to use (B.6), (B.8) and express $f\,\rho$ through ϕ and its derivatives. Then, a part of the zero mode of $f\,\rho$ vanishes due to the relation $\phi'^{\,3} + 2\phi''\,\phi'\,\phi = (\phi'^{\,2}\,\phi)'$. The same is valid for the zero mode of $\bar{f}\,\bar{\rho}$. The rest parts of the integrals $\int_0^{2\pi} \mathrm{d}z\;\, f(z)\rho(z)$ and $\int_0^{2\pi} \mathrm{d}\bar{z}\;\, \bar{f}(\bar{z})\bar{\rho}(\bar{z})$ coincide trivially.

C Useful relations in Liouville theory

The exponential $V = e^{\frac{\alpha}{2}}$ has some remarkable properties in Liouville theory. The conformal weight of V is $-\frac{1}{2}$. The Liouville equation (2.38) is equivalent to the quadratic relation for V

$$V\,\partial\bar{\partial}V - \partial V\bar{\partial}V = -\frac{\lambda\bar{\lambda}}{2}\,. \tag{C.1}$$

In addition, V fulfills the following linear relations

$$\partial^2 V(z,\bar{z}) = T(z)\,V(z,\bar{z})\,, \qquad \bar{\partial}^2 V(z,\bar{z}) = \bar{T}(\bar{z})\,V(z,\bar{z})\,, \tag{C.2}$$

where

$$T = \frac{1}{4}\,(\partial\alpha)^2 + \frac{1}{2}\,\partial^2\alpha\,, \qquad \bar{T} = \frac{1}{4}\,\left(\bar{\partial}\alpha\right)^2 + \frac{1}{2}\,\bar{\partial}^2\alpha\,, \tag{C.3}$$

are chiral ($\bar{\partial}T = 0$) and antichiral ($\partial\bar{T} = 0$) components of the stress tensor of Liouville theory.

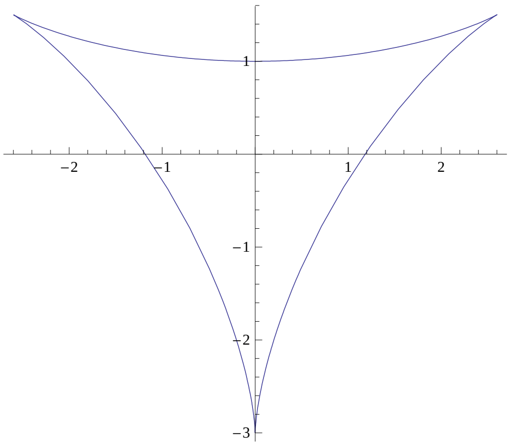

Figure 1. $n = 2$, $\bar{n} = -1$

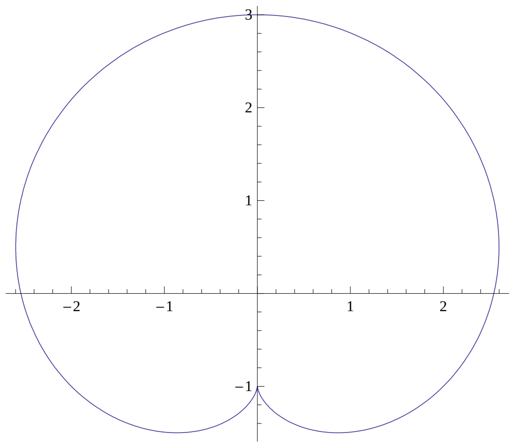

Figure 2. $n = 2$, $\bar{n} = 1$

The linear equation (C.2) defines the representation $V = A\left|\psi(z)\bar{\psi}(\bar{z}) + \chi(z)\bar{\chi}(\bar{z})\right|$, where A is a normalization constant and the functions $\psi(z)$, $\chi(z)$ and $\bar{\psi}(\bar{z})$, $\bar{\chi}(\bar{z})$ are solutions of the chiral and antichiral Hill equations, respectively:

$$\Psi''(z) = T(z)\,\Psi(z)\,, \qquad \bar{\Psi}''(\bar{z}) = \bar{T}(\bar{z})\,\bar{\Psi}(\bar{z})\,. \tag{C.4}$$

Choosing the Wronskians of these equations by

$$\psi(z)\chi'(z) - \psi'(z)\chi(z) = \epsilon\,, \quad \bar{\psi}(\bar{z})\bar{\chi}'(\bar{z}) - \bar{\psi}'(\bar{z})\bar{\chi}(\bar{z}) = -\bar{\epsilon} \tag{C.5}$$

($\epsilon = \text{sign}\lambda$, $\bar{\epsilon} = \text{sign}\bar{\lambda}$), one can fix the normalization constant A from (C.1), and obtain

$$e^{\frac{\alpha}{2}} = \sqrt{\frac{|\lambda\bar{\lambda}|}{2}}\,\left|\psi(z)\bar{\psi}(\bar{z}) + \chi(z)\bar{\chi}(\bar{z})\right|\,. \tag{C.6}$$

Introducing

$$F(z) = \epsilon\,\frac{\chi(z)}{\psi(z)}\,, \qquad \bar{F}(\bar{z}) = \bar{\epsilon}\,\frac{\bar{\psi}(\bar{z})}{\bar{\chi}(\bar{z})}\,, \tag{C.7}$$

one finds from (C.5)

$$F'(z) = \frac{1}{\psi^2(z)}\,, \qquad \bar{F}'(\bar{z}) = \frac{1}{\bar{\chi}^2(\bar{z})}\,. \tag{C.8}$$

Then, (C.6) takes the Liouville form of the general solution (2.39).

D Spiky string configurations in $\mathbb{R}^{1,2}$

In this appendix we present six pictures (see figures 1–6) of spiky string configurations at a fixed time. They are constructed by eq. (2.62) for different values of n and $\bar{n}$, with $\Lambda = |n\bar{n}|$. This value of the scale factor Λ is chosen just for convenience. The values on n and $\bar{n}$ are indicated below the pictures. The left pictures correspond to $n\bar{n} < 0$ and the rights to $n\bar{n} > 0$. These pictures demonstrate the properties 1-5, discussed in subsection 2.3.

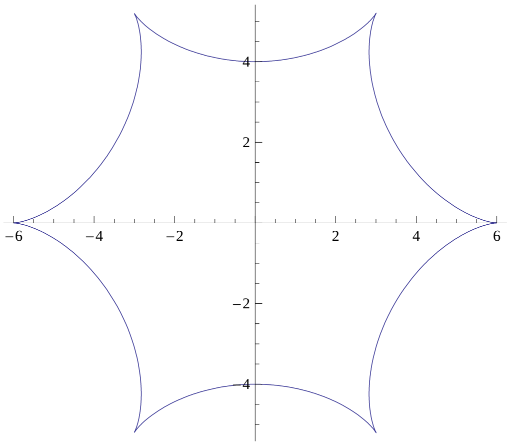

Figure 3. $n = 5$, $\bar{n} = -1$

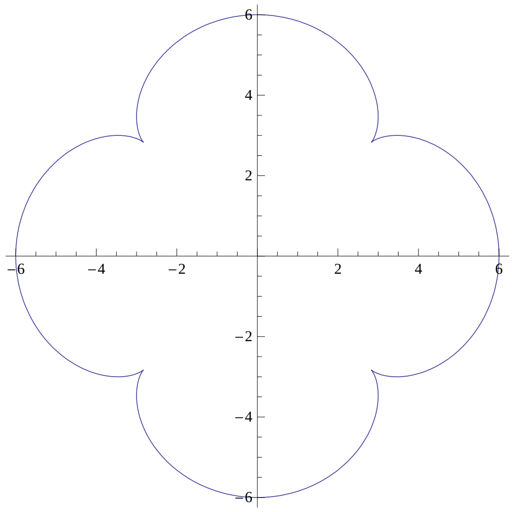

Figure 4. $n = 5$, $\bar{n} = 1$

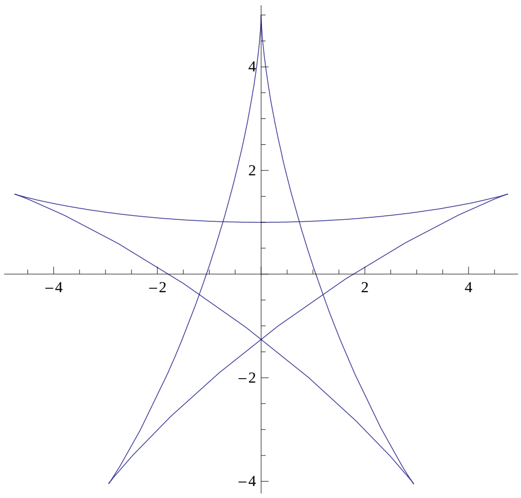

Figure 5. $n = 3$, $\bar{n} = -2$

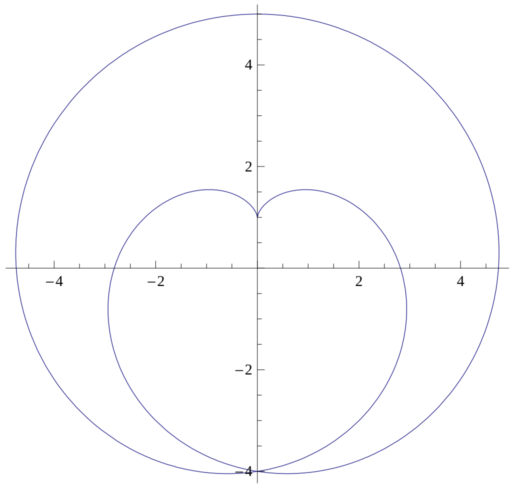

Figure 6. $n = 3$, $\bar{n} = 2$

E Properties of the spiky strings in $\mathbb{R}^{1,2}$

In this appendix we continue the discussion on spiky string surfaces in $\mathbb{R}^{1,2}$ and give a proof of the properties 3, 4 and 5 presented in subsection 2.3.

First note that the normal vector (2.41) can be written in the form (2.17), with the conformal factor (2.39) and B, $\bar{B}$ given by (2.40). Eqs. (2.53) and (2.57) then yield

$$N = \frac{1}{\cos\varphi_-}\begin{pmatrix} \sin\varphi_- \\ -\sin\varphi_+ \\ \cos\varphi_+ \end{pmatrix}, \qquad \text{(E.1)}$$

where φ_+ and φ_- are the following variables

$$\varphi_\pm = \frac{1}{2}(nz \mp \bar{n}\bar{z}) = \frac{1}{2}(n \mp \bar{n})\tau + \frac{1}{2}(n \pm \bar{n})\sigma\,. \qquad \text{(E.2)}$$

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The spikes (2.58) obviously correspond to $\cos\varphi_- = 0$ (or $|\sin\varphi_-| = 1$). Therefore, the normal vector (E.1) is singular at the spikes. Having the unit norm, this vector diverges in the lightlike direction as in the lightcone gauge.

The projection of the string surface (2.54) on the (X_1, X_2)-plane can be written as

$$\vec{X}(\tau,\sigma) = \Lambda\,\frac{n+\bar{n}}{n\bar{n}}\,\cos\varphi_-\begin{pmatrix}\sin\varphi_+\\ -\cos\varphi_+\end{pmatrix} - \Lambda\,\frac{n-\bar{n}}{n\bar{n}}\,\sin\varphi_-\begin{pmatrix}\cos\varphi_+\\ \sin\varphi_+\end{pmatrix}. \tag{E.3}$$

Here $\vec{X} = (X_1, X_2)$ denotes a 2d vector and $\varphi_\pm$ are the variables (E.2). The module square of the Euclidean vector (E.3) is given by

$$\vec{X}\cdot\vec{X} = \frac{\Lambda^2}{n^2\,\bar{n}^2}\left[n^2 + \bar{n}^2 + 2n\bar{n}\cos(2\varphi_-)\right]. \tag{E.4}$$

Since the spikes (2.58) correspond to $\cos(2\varphi_-) = -1$, they rotate around the origin on a circle of the radius (2.64). The differentiation of (E.3) yields

$$\partial_\tau\vec{X} = 2\Lambda\,\sin\varphi_-\begin{pmatrix}-\sin\varphi_+\\ \cos\varphi_+\end{pmatrix}, \qquad \partial_\sigma\vec{X} = 2\Lambda\,\cos\varphi_-\begin{pmatrix}\cos\varphi_+\\ \sin\varphi_+\end{pmatrix}. \tag{E.5}$$

Expanding the tangent vector $\partial_\sigma\vec{X}(\tau,\sigma)$ for a fixed τ near to a spike and taking into account that $\cos\varphi_- = 0$ at the spikes, from (E.3) we find

$$\partial_\sigma\vec{X}(\tau,\sigma_m + \delta\sigma) = n\bar{n}\,\vec{X}(\tau,\sigma_m)\;\delta\sigma + O(\delta\sigma)^2\,, \tag{E.6}$$

where σ_m is a solution of (2.58) for a fixed τ and $\delta\sigma$ is an infinitesimal variation. Thus, the vector $\partial_\sigma X(\tau,\sigma)$ inverts its direction at $\sigma = \sigma_m$, which indicates on the spiky character of the singularity. Due to (E.6), the spike at σ_m and the radius vector $\vec{X}(\tau,\sigma_m)$ have the same or opposite directions, depending on the sign of $n\bar{n}$, as it is stated in the item 3.

Let's consider the curve $\vec{X}(\tau,\sigma)$ as a function of σ for a given τ. Its curvature is defined by the normal (to $\partial_\sigma\vec{X}$) component of $\partial^2_{\sigma\sigma}\vec{X}$

$$\partial^2_{\sigma\sigma}\vec{X}|_N = \partial^2_{\sigma\sigma}\vec{X} - \frac{\partial^2_{\sigma\sigma}\vec{X}\cdot\partial_\sigma\vec{X}}{\partial_\sigma\vec{X}\cdot\partial_\sigma\vec{X}}\;\partial_\sigma\vec{X}\,. \tag{E.7}$$

The curvature with respect to the origin is positive if the scalar product $\partial^2_{\sigma\sigma}\vec{X}|_N\cdot\vec{X}$ is negative and vice versa. Calculating the vector $\partial^2_{\sigma\sigma}\vec{X}$ from (E.5) and using (E.7), we obtain

$$\partial^2_{\sigma\sigma}\vec{X}|_N\cdot\vec{X} = -\Lambda^2\,\frac{(n-\bar{n})^2}{n\,\bar{n}}\;\cos^2\varphi_-\,. \tag{E.8}$$

This equation proves the property 4 for an arbitrary τ.

Finally, considering the property 5, we calculate the differential of the polar angle for the curve (2.62) and integrate it in the interval $[\sigma_m, \sigma_{m+1}]$ defined by (2.63). This provides the rotation angle

$$\Delta\phi = \frac{|n\bar{n}(n+\bar{n})|}{|n-\bar{n}|}\int_0^{2\pi} d\theta\;\frac{1+\cos\theta}{n^2+\bar{n}^2+2n\bar{n}\cos\theta}\,, \tag{E.9}$$

where $\theta = |n-\bar{n}|\sigma$. Writing (E.9) as a contour integral over the unit circle with $\zeta = e^{i\theta}$, and calculating the residues of the integrand at the poles inside the unit disk we obtain (2.66). Note that the integrand has three poles at $\zeta_1 = 0$, $\zeta_2 = -n/\bar{n}$, $\zeta_3 = -\bar{n}/n$ and only two of them are inside the unit disk.

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F The Pohlmeyer scheme in $\mathbb{R}^{1,n}$

The generalization of the Pohlmeyer scheme to a higher dimensional Minkowski space $\mathbb{R}^{1,n}$ is given by the linear system

$$\begin{aligned} \partial B &= \partial\alpha\, B + u_b\, N_b\,, & \bar\partial B &= 0\,, \\ \partial\bar B &= 0\,, & \bar\partial\bar B &= \bar\partial\alpha\,\bar B + \bar u_b\, N_b\,, \\ \partial N_a &= e^{-\alpha}\, u_a\,\bar B + A_{ab}\, N_b\,, & \bar\partial N_a &= e^{-\alpha}\,\bar u_a\, B + \bar A_{ab}\, N_b\,. \end{aligned} \tag{F.1}$$

Here $B = \partial X$, $\bar B = \bar\partial X$ $\ (X\in\mathbb{R}^{1,n})$ and N_a $(a = 2,\dots,n)$ form an orthonormal basis in the normal space to the string surface

$$N_a\cdot N_b = \delta_{ab}\,, \qquad B\cdot N_a = 0 = \bar B\cdot N_a\,; \tag{F.2}$$

$u_a = \partial^2 X\cdot N_a$, $\bar u_a = \bar\partial^2 X\cdot N_a$ correspond to nonzero elements of the second quadratic forms and A_{ab}, $\bar A_{ab}$ are the torsion coefficients

$$A_{ab} = \partial N_a\cdot N_b\,, \qquad \bar A_{ab} = \bar\partial N_a\cdot N_b\,. \tag{F.3}$$

The consistency conditions for the linear system (F.1) are

$$\bar\partial\partial\alpha + e^{-\alpha}\, u_b\,\bar u_b = 0\,, \tag{F.4}$$

$$\partial\bar u_a = A_{ab}\,\bar u_b\,, \qquad \bar\partial u_a = \bar A_{ab}\, u_b\,, \tag{F.5}$$

$$\bar\partial A_{ab} - \partial\bar A_{ab} + \left[A,\bar A\right]_{ab} = e^{-\alpha}(\bar u_a u_b - \bar u_b u_a)\,. \tag{F.6}$$

The choice of the normal vectors N_a is fixed only up to a $(z,\bar z)$-dependent $O(n-1)$ transformation, which effects $u, \bar u$ and $A, \bar A$ as

$$\begin{aligned} u_a &\mapsto \Omega_{ab}\, u_b\,, & \bar u_a &\mapsto \Omega_{ab}\,\bar u_b \\ A_{ab} &\mapsto \left(\Omega A\Omega^{-1} + \partial\Omega\ \Omega^{-1}\right)_{ab}\,, & \bar A_{ab} &\mapsto \left(\Omega\bar A\Omega^{-1} + \bar\partial\Omega\ \Omega^{-1}\right)_{ab}\,. \end{aligned} \tag{F.7}$$

One can use this gauge freedom to simplify eqs. (F.4)–(F.6), find their solution and then integrate the linear system (F.1). Starting with the gauge $\bar A = 0$, as in [10] for the AdS spaces, we get from (F.5) $\bar\partial u = 0$. The consistency conditions (F.4)–(F.6) then are solved by the following parameterization

$$u_a = f_a'\,, \qquad \bar u_a = \bar f_a' - \frac{2R_a\, R_b\,\bar f_b'}{|\bar f - f|^2}\,, \qquad e^\alpha = \frac{1}{2}\,|\bar f - f|^2\,, \qquad A_{ab} = 2\,\frac{R_a\, f_b' - R_b\, f_a'}{|\bar f - f|^2}\,, \tag{F.8}$$

with $R_a = \bar f_a - f_a$ and $f_a = f_a(z)$, $\bar f_a = \bar f_a(\bar z)$. This parameterization leads to

$$N_a = (\delta_{ab} + e^{-\alpha}\, R_a\, f_b)\mathbf{e}_b + e^{-\alpha}\, R_a\,\mathbf{e}_+ + (2f_a + e^{-\alpha}\,|f|^2\, R_a)\mathbf{e}_- \tag{F.9}$$

$$B = f_a\,\mathbf{e}_a + \mathbf{e}_+ + |f|^2\,\mathbf{e}_-\,, \qquad \bar B = \bar f_a\,\mathbf{e}_a + \mathbf{e}_+ + |\bar f|^2\,\mathbf{e}_-\,, \tag{F.10}$$

where $\mathbf{e}_+, \mathbf{e}_-, \mathbf{e}_a$ is a generalization of the basis (2.15) to $\mathbb{R}^{1,n}$ and $e^{-\alpha}$ is given by (F.8). The tangent vectors (F.10) apparently correspond to the lightcone gauge strings.

G Useful formulae in $SL(2,\mathbb{R})$

Here we present some useful formulas for $sl(2,\mathbb{R})$ algebra and $\mathrm{SL}(2,\mathbb{R})$ group.

From eq. (3.7) follows $a^2 = \langle a\,a\rangle\, I$, for any $a \in sl(2,\mathbb{R})$. In particular, if $\langle a\,a\rangle = 0$, then a is nilpotent $a^2 = 0$. Eq. $a^2 = \langle a\,a\rangle\, I$ helps to find a compact form of e^a

$$\begin{aligned}
&e^a = \cosh\theta\; I + \sinh\theta\;\hat{a}\,, && \text{with}\theta = \sqrt{\langle\, a\,a\,\rangle}\,, && \hat{a} = \frac{a}{\theta}\,, && \text{if}\langle\, a\,a\,\rangle > 0\,;\\
&e^a = \cos\theta\;\; I \,+\, \sin\theta\;\hat{a}\,, && \text{with}\theta = \sqrt{-\langle\, a\,a\,\rangle}\,, && \hat{a} = \frac{a}{\theta}\,, && \text{if}\langle\, a\,a\,\rangle < 0\,;\\
&e^a = I + a\,, && && && \text{if}\langle\, a\,a\,\rangle = 0\,.
\end{aligned} \tag{G.1}$$

From these equations follows that

$$\begin{aligned}
&\langle\, e^a\rangle > 1\,, && \text{if}\quad \langle\, a\,a\,\rangle > 0\,; && (\text{spacelike}\;\; a\;\; \text{and hyperbolic}\;\; e^a)\\
&\langle\, e^a\rangle \in (-1,1)\,, && \text{if}\quad \langle\, a\,a\,\rangle < 0\,; && (\text{timelike}\;\; a\;\; \text{and elliptic}\;\; e^a)\\
&\langle\, e^a\,\rangle = 1\,, && \text{if}\quad \langle\, a\,a\,\rangle = 0;. && (\text{lightlike}\;\; a\;\; \text{and parabolic}\;\; e^a)
\end{aligned} \tag{G.2}$$

The $SO(2)$ subgroup of $\mathrm{SL}(2,\mathbb{R})$ given by

$$e^{\theta\, t_0} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{G.3}$$

defines the rotations of t_1 and t_2 under the transformations of the adjoint representation

$$e^{\frac{1}{2}\,\theta\, t_0}\, t_1\, e^{-\frac{1}{2}\,\theta\, t_0} = t_1\, \cos\theta + t_2\, \sin\theta\,, \qquad e^{\frac{1}{2}\,\theta\, t_0}\, t_2\, e^{-\frac{1}{2}\,\theta\, t_0} = t_2\, \cos\theta - t_1\, \sin\theta\,. \tag{G.4}$$

Other useful relations are

$$(t_1 + it_2)\, e^{\theta\, t_0} = e^{i\theta}\, (t_1 + it_2) = e^{-\theta\, t_0}\, (t_1 + it_2)\,, \tag{G.5}$$

$$(I - it_0)\, e^{\theta\, t_0} \;=\; e^{i\theta}\, (I - it_0) \;=\; e^{\theta\, t_0}\;\, (I - it_0)\,. \tag{G.6}$$

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